

Classification

HMTH407/HFM307

September 2022

Find the general solution of

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$$

Solution:

In this case, $a = 4$, $2b = 5$ and $c = 1$. So that

$$b^2 - ac = \left(\frac{5}{2}\right)^2 - (4)(1) = \frac{25}{4} - 4 > 0.$$

Hence the equation is hyperbolic. The characteristic equations are obtained from

$$\frac{dy}{dx} = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{4} = 1, \text{ or } \frac{1}{4}.$$

The characteristic equations are

$$\xi = y - x, \quad \eta = y - \frac{1}{4}x.$$

Using these curves, the original equation is transformed as follows:

$$u_x = w_\xi \xi_x + w_\eta \eta_x, \quad u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{xx} = w_{\xi\xi} \xi_x^2 + 2w_{\xi\eta} \xi_x \eta_x + w_{\eta\eta} \eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx}$$

$$u_{yy} = w_{\xi\xi} \xi_y^2 + 2w_{\xi\eta} \xi_y \eta_y + w_{\eta\eta} \eta_y^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy}$$

$$u_{xy} = w_{\xi\xi} \xi_x \xi_y + w_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + w_{\eta\eta} \eta_x \eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}$$

In this case

$$\xi_x = -1, \quad \xi_y = 1, \quad \eta_x = -\frac{1}{4}, \quad \eta_y = 1$$

and

$$\xi_{xx} = \xi_{yy} = \xi_{xy} = 0,$$

$$\eta_{xx} = \eta_{yy} = \eta_{xy} = 0.$$

Hence,

$$\begin{aligned}
u_{xx} &= w_{\xi\xi} + \frac{1}{2}w_{\xi\eta} + \frac{1}{16}w_{\eta\eta} \\
u_{yy} &= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \\
u_{xy} &= -w_{\xi\xi} - \frac{5}{4}w_{\xi\eta} - \frac{1}{4}w_{\eta\eta}.
\end{aligned}$$

Combining, we have

$$\begin{aligned}
4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y &= 4(w_{\xi\xi} + \frac{1}{2}w_{\xi\eta} + \frac{1}{16}w_{\eta\eta}) \\
&\quad + 5(-w_{\xi\xi} - \frac{5}{4}w_{\xi\eta} - \frac{1}{4}w_{\eta\eta}) \\
&\quad + w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \\
&\quad - w_{\xi} - \frac{1}{4}w_{\eta} + w_{\xi} + w_{\eta} \\
&= -\frac{9}{4}w_{\xi\eta} - w_{\xi} - \frac{1}{4}w_{\eta} + w_{\xi} + w_{\eta} \\
&= -\frac{9}{4}w_{\xi\eta} + \frac{3}{4}w_{\eta}
\end{aligned}$$

The original equation is transformed into

$$w_{\xi\eta} = \frac{1}{3}w_{\eta} - \frac{8}{9}.$$

Let $v = w_{\eta}$. Then

$$v_{\xi} = \frac{1}{3}v - \frac{8}{9}.$$

We solve this ODE using integrating factors. Multiplying the ODE by the integrating factor $e^{-\frac{1}{3}\xi}$, we get:

$$e^{-\frac{1}{3}\xi}v_{\xi} - \frac{1}{3}e^{-\frac{1}{3}\xi}v = -\frac{8}{9}e^{-\frac{1}{3}\xi}.$$

Using the product rule, we get

$$\frac{\partial}{\partial \xi} \left(e^{-\frac{1}{3}\xi}v \right) = -\frac{8}{9}e^{-\frac{1}{3}\xi}.$$

Integrating with respect to ξ ,

$$e^{-\frac{1}{3}\xi}v = -\frac{8}{9} \int e^{-\frac{1}{3}\xi} d\xi = \frac{8}{3}e^{-\frac{1}{3}\xi} + g(\eta).$$

So that

$$v(\xi, \eta) = \frac{8}{3} + g(\eta)e^{\frac{1}{3}\xi}.$$

Since $v = w_\eta$, integrate v w.r.t η to get w

$$w(\xi, \eta) = \frac{8}{3}\eta + e^{\frac{1}{3}\xi}G(\eta) + F(\xi)$$

where

$$G(\eta) = \int g(\eta) d\eta$$

Substituting $\xi = y - x$ and $\eta = y - \frac{1}{4}x$, we obtain

$$u(x, t) = \frac{8}{3} \left(y - \frac{1}{4}x \right) + e^{\frac{1}{3}(y-x)} G \left(y - \frac{1}{4}x \right) + F(y - x)$$