UNIVERSITY OF ZIMBABWE

BSc Honours in Mathematics & Computational Sciences: Level 4

PARTIAL DIFFERENTIAL EQUATIONS 1 (HMTH407)

September/October 2022
Time : 2 hours

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A5.

A1. Find the general solution of the partial differential equation

$$(1 + x^2)u_x + u_y = 0$$

and sketch some of the characteristic curves. [8]

A2. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous. Provide reasons.

(a)

$$\sqrt{1 + x^2}(\cos y)u_x + u_{xxy} - [\arctan(x/y)]u = 0$$

[4]

(b)

$$u_t + u_{xx} + \sqrt{1 + u} = 0$$

[4]

A3. Consider the one-dimensional wave equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \\ u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x). \end{aligned}$$

Use an energy argument to show that if $u(x, t)$ and $v(x, t)$ solve the wave equation, then $u(x, t) = v(x, t)$. [8]

A4. Consider the heat equation

$$u_t - k u_{xx} = 0$$
$$u(x,0) = \phi(x)$$

Find a solution to the heat equation with

$$\phi(x,0) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Leave your solution in terms of the error function. [8]

A5. Solve

$$u_{xx} + u_{yy} = 0$$

in the disk $\{r < a\}$ with boundary condition

$$u(a,\theta) = 1 + 3\sin\theta, \text{ on } r = a.$$

[8]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B6 to B9.

B6. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - 5\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - 1 = 0$$

(a) Classify the equation as elliptic/parabolic/hyperbolic.

(b) By a change of variables, reduce the partial differential equation to canonical form.

(c) Hence, find the general solution. [5, 15, 10]

B7. Consider the one-dimensional linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \quad c > 0.$$

(a) By using the method of characteristics, show that the general solution of the wave equation is given by

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are arbitrary.

(b) The wave equation is supplemented by the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Show that the solution is given by d'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

(c) Assuming that $f, g \in C^2$, verify that the solution in Part (b) satisfies the wave equation and the initial conditions. [10,15,5]

B8. (a) The transverse displacement of an elastic bar is given by the fourth order initial value problem

$$\begin{aligned}
&\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \quad -\infty < x < \infty, \quad t > 0 \\
&u(x,0) = f(x), \\
&u_t(x,0) = 0.
\end{aligned}$$

Use the method of Fourier Transforms to find a solution $u(x,t)$ to the fourth order IVP.

(b) Use Laplace Transforms to solve the heat equation

$$u_t - u_{xx} = 0, \quad 0 < x < \infty, \quad t > 0$$

directly with the conditions

$$u(x,0) = 0, \quad u(0,t) = f(t), \quad u(x,t) \text{ is bounded }.$$

Hint:

$$\mathcal{L}^{-1}\left(e^{-\sqrt{s}x}\right) = \frac{x}{\sqrt{4\pi t^3}} e^{\left(-x^2/4t\right)}$$

[15, 15]

B9. Consider the Laplace equation in the upper half plane

$$\begin{aligned}
&u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad y > 0 \\
&u(x,0) = f(x), \quad x \in \mathbb{R}
\end{aligned}$$

(a) Assuming the solution stays bounded as $y \to \infty$ use a Fourier Transform to show that the solution $u(x,y)$ is given by

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{(x-\tau)^2 + y^2}\, d\tau.$$

(b) Show that the solution to the bounded Neumann problem

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad y > 0$$
$$u_y(x,0) = g(x), \quad x \in \mathbb{R}$$

is given by

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x-\xi)\ln(y^2+\xi^2)\, d\xi + C$$

Hint: You may use a coordinate tranformation $w = u_y$ and reduce to a Dirichlet problem.

[15, 15]

LIST OF FORMULAS

(a) **Change of variables**

Let $u(x,y) = w(\xi(x,y), \eta(x,y))$.

$$u_x = w_\xi \xi_x + w_\eta \eta_x$$

$$u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{xx} = w_{\xi\xi}\xi_x^2 + w_{\eta\eta}\eta_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_\xi\xi_{xx} + w_\eta\eta_{xx}$$

$$u_{yy} = w_{\xi\xi}\xi_y^2 + w_{\eta\eta}\eta_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_\xi\xi_{yy} + w_\eta\eta_{yy}$$

$$u_{xy} = w_{\xi\xi}\xi_x\xi_y + w_{\eta\eta}\eta_x\eta_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_\xi\xi_{xy} + w_\eta\eta_{xy}$$

(b) **Fourier Transforms**

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{i\xi x}\, dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{-i\xi x}\, d\xi$$

$$f(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1, \end{cases} \qquad \widehat{f}(\xi) = 2\frac{\sin(\xi)}{\xi}$$

$$f(x) = \frac{1}{2}e^{-|x|}, \quad \widehat{f}(\xi) = \frac{1}{1+\xi^2}$$

$$f(x) = e^{-ax^2}, \qquad \widehat{f}(\xi) = \sqrt{\frac{\pi}{a}}e^{-\xi^2/(4a)}$$

$$\mathcal{F}(f^{(n)}(x)) = (-i\xi)^n\mathcal{F}(f(x))$$

END OF QUESTION PAPER