UNIVERSITY OF ZIMBABWE

MTSCS546

MTSCS546 (MSc in Computational Science and Mathematical Modelling)

NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

November/December 2022
Time : 3 hours

Candidates may attempt AT MOST FOUR questions being careful to number them A1 to A6.
Total marks: 100

A1. Consider the advection equation $u_t + au_x = 0,\ a > 0$. Analyse the stability of the following difference schemes.

(a) forward difference: $v_j^{n+1} = v_j^n - a\mu\left(v_{j+1}^n - v_j^n\right)$ [7]

(b) the Lax Friedrichs scheme: $v_j^{n+1} = \frac{1}{2}\left(v_{j-1}^n + v_{j+1}^n\right) - \frac{a\mu}{2}\left(v_{j+1}^n - v_{j-1}^n\right).$ [7]

(c) the leapfrog scheme: $v_j^{n+1} = v_j^{n-1} - a\mu\left(v_{j+1}^n - v_{j-1}^n\right)$ [7]

where $\mu := \Delta t/\Delta x$.

(d) Suppose that the leapfrog scheme is used to approximate the advection equation

$$u_t + 5u_x = 0$$

with an initial condition $u(x, 0) = u_0(x)$ and subject to periodic boundary conditions on a unit interval $[0, 1]$. If the unit interval is sub-divided uniformly by a mesh-size of $h = 0.01$, what is the maximum possible value of the time-step $k = \Delta t$ so that the scheme remains stable? [4]

A2. The scalar diffusion equation

$$u_t = \nu u_{xx}$$

with $\nu > 0$, is discretized explicitly

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \nu\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2}.$$

(a) Compute the truncation error of the method. [12]

(b) What is the order $\mathcal{O}\left[(\Delta t)^p + (\Delta x)^q\right]$ of the method? [3]

(c) Analyse its stability. [10]

A3. The general form of a difference scheme for approximating a well-posed PDE is given by

$$B_1 v^{n+1} = B_0 v^n + F^n,$$

where B_1, B_0 are difference operators and $v^n = \{v_j^n : j = 0, 1, \cdots, J\}$ where J is the total number of mesh points and $v_j^n \approx u(x_j, t_n)$

(a) Explain the meaning of the terms: consistency, order of accuracy, stability, and convergence of the difference scheme. [4]

(b) Suppose that the difference scheme is consistent. Prove that stability implies convergence. In your proof, indicate places where stability and consistency are used to demonstrate convergence. [15]

(c) Consider the forward in time centered in space discretization of the advection equation $u_t + au_x = 0$:

$$v_j^{n+1} = v_j^n - \frac{a\mu}{2}\left(v_{j+1}^n - v_{j-1}^n\right).$$

Does this scheme converge to the solution of the advection equation? Explain in terms of Theorem A3 (b). [6]

A4. The diffusion equation

$$u_t = u_{xx} + u_{yy}$$

on a square is to be approximated by the Alternating Directions Implicit difference scheme.

(a) Explain some advantages of the ADI method over the Crank-Nicolson method for the two-dimensional diffusion equation. [5]

(b) The locally one-dimensional ADI scheme in two-dimensions is

$$\left(1 - \frac{1}{2}\mu_x\delta_x^2\right) v^{n+1/2} = \left(1 + \frac{1}{2}\mu_y\delta_y^2\right) v^n$$

$$\left(1 - \frac{1}{2}\mu_y\delta_y^2\right) v^{n+1} = \left(1 + \frac{1}{2}\mu_x\delta_x^2\right) v^{n+1/2}$$

where

$$\mu_x = \frac{\Delta t}{(\Delta x)^2} \text{ and } \delta_x^2 = v_{j+1} - 2v_j + v_{j-1}.$$

By eliminating the intermediate variable $v^{n+1/2}$ verify that the ADI scheme is a modification of the Crank-Nicolson scheme up to a higher order term.

$$\left(1 - \frac{1}{2}\mu_x\delta_x^2 - \frac{1}{2}\mu_y\delta_y^2\right) v^{n+1} = \left(1 + \frac{1}{2}\mu_x\delta_x^2 + \frac{1}{2}\mu_y\delta_y^2\right) v^n.$$

[8]

(c) Show that the ADI scheme in Part (4b) is unconditionally stable. [12]

A5. (a) Calculate the dispersion relation for the two dimensional Schrödinger equation

$$u_t = i\Delta u$$

where $\Delta u = u_{xx} + u_{yy}$.
Hint: Consider a plane wave solution of the form $u = e^{i(\xi_x x + \xi_y y - \omega t)}$ [4]

(b) Show that the dispersion relation of the Crank-Nicolson approximation of the Schrödinger equation

$$\left(1 - i\mu\frac{1}{2}\delta_x^2 - i\mu\frac{1}{2}\delta_y^2\right) v^{n+1} = \left(1 + i\mu\frac{1}{2}\delta_x^2 + i\mu\frac{1}{2}\delta_y^2\right) v^n$$

where

$$\Delta x = \Delta y = h, \quad k = \Delta t, \quad \text{and } \mu = \frac{k}{h^2}$$

is given by

$$\tan\left(\frac{\omega k}{2}\right) = 2\mu\left[\sin^2\left(\frac{\xi_x h}{2}\right) + \sin^2\left(\frac{\xi_y h}{2}\right)\right]$$

Hint: Consider a plane wave solution of the form $u = e^{i(\xi_x jh + \xi_y \ell h - \omega t)}$ [10]

(c) Compute an approximate dispersion relation of the Crank-Nicolson scheme for $|\xi_x|h \ll 1, \ \ |\xi_y|h \ll 1$ and compare with the dispersion relation of the Schrödinger equation from Part 5(a). [4]

(a) For a dispersion relation $\omega = \omega(\xi_x, \xi_y)$ in two-dimensions, define the **phase velocity** and **group velocity**. [2]

(d) Calculate the group velocity for the Crank Nicolson scheme of the Schrödinger equation. Comment on the value of the group velocity for $|\xi_x|h \ll 1, \ \ |\xi_y|h \ll 1$ and for $\xi_x h \approx \xi_y h \approx \pi$. [5]

A6. Suppose that the mesh points are chosen such that

$$0 = x_0 < x_1 < x_2 < \cdots < x_{J-1} < x_J = 1$$

but are otherwise arbitrary for some J representing the number of sub-divisions. The heat equation $u_t = u_{xx}$ is approximated over the interval $0 \le t \le t_f$ by

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{2}{\Delta x_{j-1} + \Delta x_j}\left(\frac{v_{j+1}^n - v_j^n}{\Delta x_j} - \frac{v_j^n - v_{j-1}^n}{\Delta x_{j-1}}\right)$$

where $\Delta x_j = x_{j+1} - x_j$.

(a) Show that the leading terms of the truncation error of this approximation are

$$\begin{aligned} T_j^n &= \frac{1}{2}\Delta t\, u_{tt} - \frac{1}{3}(\Delta x_j - \Delta x_{j-1})u_{xxx} \\ &\quad - \frac{1}{12}\left[(\Delta x_j)^2 + (\Delta x_{j-1})^2 - \Delta x_j \Delta x_{j-1}\right] u_{xxxx}. \end{aligned}$$

[15]

(b) Suppose now that the boundary and initial conditions $u(0,t), u(1,t),$ and $u(x,0)$ are provided. Let $\Delta x = \max \Delta x_j$ and suppose the mesh is sufficiently regular such that $|\Delta x_j - \Delta x_{j-1}| \leq \alpha(\Delta x)^2$ for every $j = 1, 2, 3, \cdots, J-1$, where $\alpha > 0$ is constant.

Show that

$$|v_j^n - u(x_j, t_n)| \leq \left(\frac{1}{2}\Delta t \, M_{tt} + (\Delta x)^2 \left\{ \frac{1}{3}\alpha M_{xxx} + \frac{1}{12}[1 + \alpha \Delta x] M_{xxxx} \right\} \right) t_f$$

provided that the stability condition

$$\Delta t \leq \frac{1}{2}\Delta x_{j-1}\Delta x_j, \quad j = 1, 2, \cdots, J-1,$$

is satisfied. [10]

END OF QUESTION PAPER