MTSCS546 Assignment 2

Numerical Methods for Partial Differential Equations

Due: 18 November, 2022

1. Suppose a well-posed linear evolutionary PDE is discretized by a difference scheme
$$B_1 v^{n+1} = B_0 v^n + F^n, \text{ for } n\Delta t \leq t_F.$$
   (a) Explain what it means for the difference scheme to be consistent, convergent and stable.
   (b) Prove that if the difference scheme is consistent with the initial-boundary value PDE, then stability implies convergence. Indicate in your proof where consistency and stability are utilized to obtain convergence.
   (c) The difference scheme
$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2} + \frac{1}{2}v_j^n$$
   is applied to discretize the following parabolic PDE
$$u_t = u_{xx} + u.$$
   Numerical results on a test example show that the difference scheme does **not** converge to the true solution. Explain this observation in light of the result from Part 1(b).

2. (a) Define dispersion, phase velocity and group velocity for a linear PDE.
   (b) Is the advection equation ($u_t + au_x = 0$) dispersive? The KdV equation ($u_t + \rho u_x + \nu u_{xxx} = 0$) ?
   (c) Calculate the dispersion relations of the following schemes for the advection equation: Lax-Wendroff, Crank-Nicolson.
   (d) What is the effect of dissipativity on dispersive numerical schemes?

3. (a) Calculate and plot the dispersion relation for the one dimensional Schrödinger equation
$$u_t = iu_{xx}$$
   (b) Calculate the dispersion relation for the Crank-Nicolson scheme applied to the Schrödinger equation.
   (c) Calculate the group velocity. Compare the group velocity of the Crank-Nicolson scheme with that of the equation $u_t = iu_{xx}$.