MTSCS546 Assignment 1

Numerical Methods for Partial Differential Equations

Due: 29 October, 2022

1. (a) For the advection equation $u_t + au_x = 0$ with $a > 0$, show that the forward difference scheme

$$v_j^{n+1} = v_j^n - a\mu\left(v_{j+1}^n - v_j^n\right)$$

is unconditionally unstable by computing the **amplification factor**.

(b) On the other hand, show that the scheme

$$v_j^{n+1} = v_j^n - a\mu\left(v_j^n - v_{j-1}^n\right)$$

is stable for $a > 0$ provided that $\mu a \leq 1$, but unstable for any $\mu \geq 0$ if $a < 0$. Explain.

(c) The **upwind** difference scheme for the advection equation is given by

$$v_j^{n+1} = v_j^n - a\mu \begin{cases} (v_{j+1}^n - v_j^n), & \text{if } a < 0, \\ (v_j^n - v_{j-1}^n), & \text{if } a \geq 0. \end{cases}$$

Show that the upwind scheme is stable if $|a|\mu \leq 1$.

2. Suppose that the mesh points are chosen such that

$$0 = x_0 < x_1 < x_2 < \cdots < x_{J-1} < x_J = 1$$

but are otherwise arbitrary for some J representing the number of subdivisions. The heat equation $u_t = u_{xx}$ is approximated over the interval $0 \leq t \leq t_f$ by

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{2}{\Delta x_{j-1} + \Delta x_j}\left(\frac{v_{j+1}^n - v_j^n}{\Delta x_j} - \frac{v_j^n - v_{j-1}^n}{\Delta x_{j-1}}\right)$$

where $\Delta x_j = x_{j+1} - x_j$.

(a) Show that the leading terms of the truncation error of this approximation are

$$\begin{aligned} T_j^n &= \frac{1}{2}\Delta t\, u_{tt} - \frac{1}{3}(\Delta x_j - \Delta x_{j-1})u_{xxx} \\ &\quad - \frac{1}{12}\left[(\Delta x_j)^2 + (\Delta x_{j-1})^2 - \Delta x_j \Delta x_{j-1}\right] u_{xxxx}. \end{aligned}$$

(b) Suppose now that the boundary and initial conditions $u(0,t), u(1,t),$ and $u(x,0)$ are provided. Let $\Delta x = \max \Delta x_j$ and suppose the mesh is sufficiently regular such that $|\Delta x_j - \Delta x_{j-1}| \leq \alpha(\Delta x)^2$ for every $j = 1, 2, 3, \cdots, J-1$, where $\alpha > 0$ is constant.

Show that

$$|v_j^n - u(x_j, t_n)| \leq \left(\frac{1}{2}\Delta t \; M_{tt} + (\Delta x)^2 \left\{ \frac{1}{3}\alpha M_{xxx} + \frac{1}{12}[1 + \alpha \Delta x] M_{xxxx} \right\} \right) t_f$$

provided that the stability condition

$$\Delta t \leq \frac{1}{2}\Delta x_{j-1}\Delta x_j, \quad j = 1, 2, \cdots, J-1,$$

is satisfied.

3. (a) Show that the leading terms in the truncation error of the Peaceman-Rachford ADI method for the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy}$$

are

$$\begin{aligned} T^{n+1/2} \quad &= \quad (\Delta t)^2 \left[\frac{1}{24} u_{ttt} - \frac{1}{8}\left(u_{xxtt} + u_{yytt}\right) + \frac{1}{4} u_{xxyyt} \right] \\ &\quad - \frac{1}{12}\left[(\Delta x)^2 u_{xxxx} + (\Delta y)^2 y_{yyyy} \right]. \end{aligned}$$

(b) Show that the Douglas-Rachford scheme

$$\begin{aligned} \left(1 - \mu_x \delta_x^2\right) v^{n+1*} &= \left(1 + \mu_y \delta_y^2 + \mu_z \delta_z^2\right) v^n \\ \left(1 - \mu_y \delta_y^2\right) v^{n+1**} &= v^{n+1*} - \mu_y \delta_y^2 v^n \\ \left(1 - \mu_z \delta_z^2\right) v^{n+1} &= v^{n+1**} - \mu_z \delta_z^2 v^n \end{aligned}$$

for the three-dimensional heat equation

$$u_t = u_{xx} + u_{yy} + u_{zz}$$

is unconditionally stable when applied to a rectilinear box.