MTSCS546 Assignment 2

Numerical Methods for Partial Differential Equations

Due: 18 November, 2022

1. Suppose a well-posed linear evolutionary PDE is discretized by a difference scheme

$$B_1 v^{n+1} = B_0 v^n + F^n, \text{ for } n\Delta t \leq t_F.$$

(a) Explain what it means for the difference scheme to be consistent, convergent and stable.

Solution:

Consistency:

Let u be the exact solution of the PDE. Define the truncation error T^n as follows:

$$T^n := B_1 u^{n+1} - B_0 u^n - F^n$$

A difference scheme is *consistent* if $\|T^n\| \to 0$ as the mesh parameters $\Delta t(h) \to 0$ along a refinement path.

Convergence:

A difference scheme is *convergent* if

$$\|u^n - v^n\| \to 0$$

as the mesh parameters $\Delta t(h) \to 0$ for any initial data u^0 for which the PDE is well-posed.

Stability:

Suppose u^n, w^n are two numerical solutions that start from different initial data u^0, w^0 but have the same right hand side F^n for every n. Then the numerical scheme is *stable* in the norm $\|\cdot\|$ if there exist a constant K such that

$$\|u^n - w^n\| \leq K\|u^0 - w^0\|.$$

Since

$$B_1(u^n - w^n) = B_0(u^{n-1} - w^{n-1}),$$

we have that

$$u^n - w^n = B_1^{-1} B_0 (u^{n-1} - w^{n-1})$$

this condition can be written as

$$\|u^n - w^n\| \leq \|(B_1^{-1} B_0)^n\| \|u^0 - w^0\|.$$

It follows that a scheme is stable if

$$\|(B_1^{-1}B_0)^n\| \le K \text{ for all } n\Delta t \le t_f$$

(b) Prove that if the difference scheme is consistent with the initial-boundary value PDE, then stability implies convergence. Indicate in your proof where consistency and stability are utilized to obtain convergence.

Solution:

Suppose u^n is the exact solution at time $t = n\Delta t$ and v^n the numerical solution. We want to find conditions that guarantee convergence of v^n to u^n in the norm $\|\cdot\|$ as $\Delta t(h) \to 0$. The definition of the truncation error T^n implies that:

$$B_1(u^{n+1} - v^{n+1}) = B_0(u^n - v^n) - T^n.$$

Setting $n = 0$ we see that

$$B_1(u^1 - v^1) = B_0(u^0 - v^0) - T^0.$$

But the initial conditions are the same so that $u^0 = v^0$. Hence

$$u^1 - v^1 = -B_1^{-1}B_0B_1^{-1}T^0,$$

and

$$u^2 - v^2 = -(B_1^{-1}B_0)^2B_1^{-1}T^0 - B_1^{-1}T^1.$$

Continuing in this fashion, for general n, we have

$$u^n - v^n = -\left[B_1^{-1}T^{n-1} + (B_1^{-1}B_0)B_1^{-1}T^{n-2} + \cdots + (B_1^{-1}B_0)^nB_1^{-1}T_0\right].$$

Taking norms on both sides

$$\|u^n - v^n\| \le \|B_1^{-1}\|\|T^{n-1}\| + \|(B_1^{-1}B_0)B_1^{-1}\|\|T^{n-2}\| + \cdots + \|(B_1^{-1}B_0)^nB_1^{-1}\|\|T^0\|.$$

The <u>stability</u> condition

$$\|(B_1^{-1}B_0)^nB_1^{-1}\| \le KK_1\Delta t$$

implies that

$$\|u^n - v^n\| \le KK_1\Delta t\sum_{m=0}^{n-1}\|T^m\|.$$

If the scheme is consistent, then

$$\|T^m\| \to 0$$

for every $m\Delta t \le t_f$ as $\Delta t(h) \to 0$. This implies that

$$\|u^n - v^n\| \to 0$$

if $\Delta t(h) \to 0$.

(c) The difference scheme

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2} + \frac{1}{2}v_j^n$$

is applied to discretize the following parabolic PDE

$$u_t = u_{xx} + u.$$

Numerical results on a test example show that the difference scheme does **not** converge to the true solution. Explain this observation in light of the result from Part 1(b).

Solution:
We observe that the scheme is inconsistent at the term $\frac{1}{2}v_j^n$ since the corresponding term in the PDE is not scaled by the factor of 1/2. To prove this observation, we need to show that the truncation error T_j^n does not converge to zero. We use Taylor series of the exact solution $u = u_j^n := u(x_j, t_n) \approx v_j^n$ at (x_j, t_n) to compute the truncation error.

$$u_j^{n+1} = u + (\Delta t)u_t + \frac{1}{2}(\Delta t)^2 u_{tt} + \cdots$$

so that

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = u_t + \frac{1}{2}(\Delta t)u_{tt} + \cdots .$$

By a similar computation,

$$\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots$$

The truncation error is

$$T_j^n := \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} - \frac{1}{2}u_j^n.$$

Grouping terms, we have

$$T_j^n = (u_t - u_{xx} - u) + \frac{1}{2}u + \frac{1}{2}(\Delta t)u_{tt} - \frac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots$$

The first term in parentheses vanishes since it is the PDE we are approximating. If Δt is chosen such that the scheme is stable, then $T_j^n \to \frac{1}{2}u_j^n$ as $\Delta t(h) \to 0$. This shows that the scheme is not consistent. Therefore, by the Lax Equivalence Theorem, the scheme is not convergent.

2. (a) Define dispersion, phase velocity and group velocity for a linear PDE.
Solution:
Consider the plane wave solution $e^{i(\xi x - \omega t)}$ of a linear PDE where ξ

is the wavenumber and ω is the frequency. The dispersion relation of the PDE is the functional relation

$$\omega = \omega(\xi)$$

between the wavenumber and the wavelength. A linear PDE is dispersive if the dispersion relationship is not linear. The phase velocity c_p is defined as the function

$$c_p := \frac{\omega}{\xi}.$$

The phase velocity measures the rate at which the wave propagates in the medium. The group velocity is defined as

$$c_g := \frac{d\omega}{d\xi}.$$

It measures the speed of wave packets.

(b) Is the advection equation ($u_t + au_x = 0$) dispersive? The KdV equation ($u_t + \rho u_x + \nu u_{xxx} = 0$) ?

Solution:

Consider a plane wave solution of the form $e^{i(\xi x - \omega t)}$. Plugging into the advection equation gives

$$-i\omega + ai\xi = 0.$$

This is equivalent to

$$\omega = a\xi.$$

Different wavelengths propagate at the same phase velocity $\frac{\omega}{\xi} = a$ hence the advection equation is not dispersive. On the other hand, the dispersion relation of the KdV equation is given by

$$-i\omega + \rho i\xi + \nu(i\xi)^3 = 0.$$

This can be simplified to give

$$\omega = \rho\xi - \nu\xi^3 = \xi(\rho - \nu\xi^2).$$

This dispersion relation is nonlinear. Hence the KdV equation is dispersive.

(c) Calculate the dispersion relations of the following schemes for the advection equation: Lax-Wendroff, Crank-Nicolson.

Solution:

The Lax-Wendroff scheme for the advection equation is given by

$$v_j^{n+1} = v_j^n - \frac{a\mu}{2}(v_{j+1}^n - v_{j-1}^n) + \frac{a^2\mu^2}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n)$$

where $\mu = k/h^2$, $k = \Delta t$ and $h = \Delta x$. We propose a discrete plane wave solution of the form $e^{i(\xi jh - \omega nk)}$. Plugging into the Lax Wendroff scheme yields

$$e^{-i\omega k} = 1 - \frac{a\mu}{2}\left(e^{i\xi h} - e^{-i\xi h}\right) + \frac{a^2\mu^2}{2}\left(e^{i\xi h} - 2 + e^{-i\xi h}\right).$$

Using

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

and

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

we get

$$e^{-i\omega k} = 1 - a\mu i \sin(\xi h) - 2a^2\mu^2 \sin^2(\xi h/2)$$

Taking the real and imaginary parts

$$\cos(\omega k) = 1 - 2a^2\mu^2 \sin^2(\xi h/2), \quad -\sin(\omega k) = -a\mu sin(\xi h).$$

Dividing, we obtain the dispersion relation for Lax Wendroff

$$\tan(\omega k) = \frac{a\mu \sin(\xi h)}{1 - 2a^2\mu^2 \sin^2(\xi h/2)}.$$

The Crank-Nicolson scheme for the advection equation is given by

$$\frac{v_j^{n+1} - v_j^n}{k} + \frac{a}{2}\left(\frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2h} + \frac{v_{j+1}^n - v_{j-1}^n}{2h}\right) = 0.$$

Let $\mu = k/h$, and $e^{i(\xi jh - \omega nk)}$ be a discrete plane wave. Plugging into the Crank-Nicolson scheme

$$e^{-i\omega k} - 1 = -\frac{a\mu i}{2}\left(\frac{e^{i\xi h} - e^{-i\xi h}}{2i}\right)e^{-i\omega k} - \frac{a\mu i}{2}\left(\frac{e^{i\xi h} - e^{-i\xi h}}{2i}\right).$$

From which we derive the expression

$$e^{-i\omega k} - 1 = -\frac{a\mu i}{2}\sin(\xi h)(e^{-i\omega k} + 1)$$

or

$$\frac{1}{i}\frac{e^{-i\omega k} - 1}{e^{-i\omega k} + 1} = -\frac{a\mu}{2}\sin(\xi h).$$

Note that the right hand side is equivalent to (show this)

$$-\tan(\omega k/2) = \frac{1}{i}\frac{e^{-i\omega k} - 1}{e^{-i\omega k} + 1}.$$

Therefore the dispersion relation of the Crank-Nicolson scheme for the advection equation is

$$\tan(\omega k/2) = \frac{a\mu}{2}\sin(\xi h)$$

(d) What is the effect of dissipativity on dispersive numerical schemes?

Solution:

Adding dissipation to dispersive schemes helps to dampen high frequency parasitic waves that emerge due to numerical dispersion.

3. (a) Calculate and plot the dispersion relation for the one dimensional Schrödinger equation

$$u_t = iu_{xx}$$

Solution:

Consider a plane wave solution $e^{i(\xi x-\omega t)}$. We get after plugging

$$-i\omega = i(i\xi)^2$$

which is equivalent to

$$\omega = \xi^2.$$

(b) Calculate the dispersion relation for the Crank-Nicolson scheme applied to the Schrödinger equation.

Solution:

The Crank-Nicolson scheme for the Schrödinger equation is

$$v^{n+1} = v^n + \frac{\mu i}{2}\left(\delta_x^2 v^{n+1} + \delta_x^2 v^n\right)$$

where

$$\delta_x^2 v^n := v_{j+1}^n - 2v_j^n + v_{j-1}^n$$

and $\mu = k/h^2$. Taking a discrete plane wave solution $e^{i(\xi jh-\omega nk)}$ we get

$$e^{-i\omega k} - 1 = \frac{\mu i}{2}\left(e^{i\xi h} - 2 + e^{-i\xi h}\right)\left(e^{-i\omega k} + 1\right)$$

which is identical to (show this)

$$\frac{1}{i}\frac{e^{-i\omega k} - 1}{e^{-i\omega k} + 1} = -2\mu \sin^2(\xi h/2).$$

Therefore the dispersion relation is

$$\tan(\omega k/2) = 2\mu \sin^2(\xi h/2)$$

(c) Calculate the group velocity. Compare the group velocity of the Crank-Nicolson scheme with that of the equation $u_t = iu_{xx}$.

Solution:

The group velocity is computed using implicit differentiation. We assume the relation $\omega = \omega(\xi)$ and use the definition

$$\frac{k}{2}\sec^2(\omega k/2)\frac{d\omega}{d\xi} = 2\mu h \sin(\xi h/2)\cos(\xi h/2)$$

so that

$$\frac{d\omega}{d\xi} = \frac{2\mu h \sin(\xi h)}{k \sec^2(\omega k/2)}.$$

When ξh is small, $\sin(\xi h) \approx \xi h$ and using $\mu = k/h^2$ and $\sec^2(\omega k) \approx 1$ for small k we see

$$\frac{d\omega}{d\xi} \approx 2\xi.$$

The group velocity of the Schrödinger equation is

$$c_g = 2\xi,$$

which we get from differentiating the dispersion relation $\omega = \xi^2$.