MTSCS546 Assignment 1

Numerical Methods for Partial Differential Equations

Due: 29 October, 2022

1. (a) For the advection equation $u_t + au_x = 0$ with $a > 0$, show that the forward difference scheme

$$v_j^{n+1} = v_j^n - a\mu\left(v_{j+1}^n - v_j^n\right)$$

is unconditionally unstable by computing the **amplification factor**.

Solution:

We consider a solution of the form $v_j^n = \lambda^n e^{i\xi jh}$ where $h = \Delta x$. Simple calculations show that

$$\begin{aligned} v_j^{n+1} &= \lambda v_j^n \\ v_{j+1}^n &= e^{i\xi h} v_j^n \end{aligned}$$

Substituting into the difference scheme, we obtain after canceling v_j^n

$$\lambda = 1 - a\mu(e^{i\xi h} - 1).$$

The scheme is stable if $|\lambda| \leq 1$, otherwise it is unstable. It is sufficient to show that $|\lambda|^2 \leq 1$ since the square is easier to compute. Note that

$$|\lambda|^2 = (\mathrm{Re}(\lambda))^2 + (\mathrm{Im}(\lambda))^2.$$

Since $e^{i\xi h} = \cos(\xi h) + i\sin(\xi h)$ and $1 - \cos(\xi h) = 2\sin^2(\xi h/2)$ we have

$$\mathrm{Re}(\lambda) = 1 - a\mu(\cos(\xi h) - 1) = 1 + a\mu(1 - \cos(\xi h)) = 1 + 2a\mu\sin^2(\xi h/2)$$

$$\mathrm{Im}(\lambda) = -a\mu\sin(\xi h).$$

Hence

$$|\lambda|^2 = (1 + 2a\mu\sin^2(\xi h/2))^2 + a^2\mu^2\sin^2(\xi h).$$

Upon expansion,

$$|\lambda|^2 = 1 + 4a\mu\sin^2(\xi h/2) + 4a^2\mu^2\sin^4(\xi h/2) + a^2\mu^2\sin^2(\xi h).$$

By the double angle identity for sine, we get

$$\sin(\xi h) = \sin(2\xi h/2) = 2\sin(\xi h/2)\cos(\xi h/2)$$

and

$$a^2\mu^2\sin^2(\xi h) = 4a^2\mu^2\sin^2(\xi h/2)\cos^2(\xi h/2).$$

Combining, we get

$$|\lambda|^2 = 1+4a^2\mu^2\sin^2(\xi h/2)\left(\sin^2(\xi h/2)+\cos^2(\xi h/2)\right)+4a\mu\sin^2(\xi h/2).$$

Thus

$$|\lambda|^2 = 1 + 4a\mu\sin^2(\xi h/2)\,(a\mu+1) \geq 1$$

since $a > 0$. Hence the scheme is unconditionally unstable.

(b) On the other hand, show that the scheme

$$v_j^{n+1} = v_j^n - a\mu\left(v_j^n - v_{j-1}^n\right)$$

is stable for $a > 0$ provided that $\mu a \leq 1$, but unstable for any $\mu \geq 0$ if $a < 0$. Explain.

Solution:

Taking $v_j^n = \lambda^n e^{i\xi jh}$ one can show that

$$\lambda = 1 - a\mu\left(1 - e^{-i\xi h}\right)$$

from which

$$\mathrm{Re}(\lambda) = 1 - a\mu\,(1-\cos(\xi h)) = 1 - 2a\mu\sin^2(\xi h/2)$$

$$\mathrm{Im}(\lambda) = -a\mu\sin(\xi h) = -2a\mu\sin(\xi h/2)\cos(\xi h/2).$$

Thus

$$|\lambda|^2 = (1-2a\mu\sin^2(\xi h/2))^2 + 4a^2\mu^2\sin^2(\xi h/2)\cos^2(\xi h/2).$$

Expanding,

$$\begin{aligned}
|\lambda|^2 &= 1 - 4a\mu\sin^2(\xi h/2) + 4a^2\mu^2\sin^4(\xi h/2) + 4a^2\mu^2\sin^2(\xi h/2)\cos^2(\xi h/2)\\
&= 1 - 4a\mu\sin^2(\xi h/2) + 4a^2\mu^2\sin^2(\xi h/2)\left[\sin^2(\xi h/2)+\cos^2(\xi h/2)\right]\\
&= 1 - 4a\mu\,(1-a\mu)\sin^2(\xi h/2)\\
&\leq 1, \text{ if } a\mu \leq 1.
\end{aligned}$$

If $\mu \geq 0$ but $a < 0$ then

$$|\lambda|^2 = 1 - 4a\mu(1-a\mu)\sin^2(\xi h/2) \geq 1,$$

and the scheme is unstable.

(c) The **upwind** difference scheme for the advection equation is given by

$$v_j^{n+1} = v_j^n - a\mu \begin{cases} (v_{j+1}^n - v_j^n), & \text{if } a < 0, \\ (v_j^n - v_{j-1}^n), & \text{if } a \geq 0. \end{cases}$$

Show that the upwind scheme is stable if $|a|\mu \leq 1$.

Solution:

Taking $v_j^n = \lambda^n e^{i\xi jh}$ as usual, if $a < 0$, then

$$|\lambda|^2 = 1 + 4a\mu(1 + a\mu)\sin^2(\xi h/2),$$

Let $\beta > 0$ be such that $a := -\beta$. Then

$$|\lambda|^2 = 1 - 4\beta\mu(1 - \beta\mu)\sin^2(\xi h/2) \leq 1, \text{ if } \beta\mu := -a\mu = |a|\mu \leq 1.$$

On the other hand if $a \geq 0$ then

$$|\lambda|^2 = 1 - 4a\mu\,(1 - a\mu)\sin^2(\xi h/2) \leq 1, \text{ if } a\mu = |a|\mu \leq 1.$$

2. Suppose that the mesh points are chosen such that

$$0 = x_0 < x_1 < x_2 < \cdots < x_{J-1} < x_J = 1$$

but are otherwise arbitrary for some J representing the number of subdivisions. The heat equation $u_t = u_{xx}$ is approximated over the interval $0 \leq t \leq t_f$ by

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{2}{\Delta x_{j-1} + \Delta x_j}\left(\frac{v_{j+1}^n - v_j^n}{\Delta x_j} - \frac{v_j^n - v_{j-1}^n}{\Delta x_{j-1}}\right)$$

where $\Delta x_j = x_{j+1} - x_j$.

(a) Show that the leading terms of the truncation error of this approximation are

$$\begin{aligned} T_j^n &= \frac{1}{2}\Delta t\, u_{tt} - \frac{1}{3}(\Delta x_j - \Delta x_{j-1})u_{xxx} \\ &\quad -\frac{1}{12}\left[(\Delta x_j)^2 + (\Delta x_{j-1})^2 - \Delta x_j \Delta x_{j-1}\right]u_{xxxx}. \end{aligned}$$

Solution:

Use Taylor expansions around $u = u_j^n = u(x_j, t_n)$

$$u_j^{n+1} = u + (\Delta t)\, u_t + \frac{1}{2}(\Delta t)^2 u_{tt} + \cdots$$

$$u_{j+1}^n = u + (\Delta x_j)\, u_x + \frac{1}{2}(\Delta x_j)^2 u_{xx} + \frac{1}{6}(\Delta x_j)^3 u_{xxx} + \frac{1}{24}(\Delta x_j)^4 u_{xxxx} + \cdots$$

$$u_{j-1}^n = u - (\Delta x_{j-1})\ u_x + \frac{1}{2}(\Delta x_{j-1})^2 u_{xx} - \frac{1}{6}(\Delta x_{j-1})^3 u_{xxx} + \frac{1}{24}(\Delta x_{j-1})^4 u_{xxxx} + \cdots .$$

Hence,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = u_t + \frac{1}{2}(\Delta t) u_{tt} + \cdots$$

and

$$\frac{u_{j+1}^n - u_j^n}{\Delta x_j} = u_x + \frac{1}{2}(\Delta x_j) u_{xx} + \frac{1}{6}(\Delta x_j)^2 u_{xxx} + \frac{1}{24}(\Delta x_j)^3 u_{xxxx} + \cdots$$

$$\frac{u_j^n - u_{j-1}^n}{\Delta x_{j-1}} = u_x - \frac{1}{2}(\Delta x_{j-1}) u_{xx} + \frac{1}{6}(\Delta x_{j-1})^2 u_{xxx} - \frac{1}{24}(\Delta x_{j-1})^3 u_{xxxx} + \cdots$$

$$\begin{aligned}\frac{u_{j+1}^n - u_j^n}{\Delta x_j} - \frac{u_j^n - u_{j-1}^n}{\Delta x_j} &= \frac{1}{2}(\Delta x_j + \Delta x_{j-1}) u_{xx} + \frac{1}{6}\left[(\Delta x_j)^2 - (\Delta x_{j-1})^2\right] u_{xxx} \\ &\quad + \frac{1}{24}\left[(\Delta x_j)^3 + (\Delta x_{j-1})^3\right] u_{xxxx} + \cdots\end{aligned}$$

Set

$$RHS := \frac{2}{\Delta x_j + \Delta x_{j-1}}\left(\frac{u_{j+1}^n - u_j^n}{\Delta x_j} - \frac{u_j^n - u_{j-1}^n}{\Delta x_{j-1}}\right).$$

Then

$$\begin{aligned} RHS &= u_{xx} + \frac{1}{3}\left(\Delta x_j - \Delta x_{j-1}\right) u_{xxx} \\ &\quad + \frac{1}{12}\left[(\Delta x_j)^2 + (\Delta x_{j-1})^2 - \Delta x_j \Delta x_{j-1}\right] u_{xxxx} + \cdots \end{aligned}$$

where we used

$$a^2 - b^2 = (a+b)(a-b) \text{ and } a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$$

Combining, we get

$$\begin{aligned} T_j^n &= (u_t - x_{xx}) + \frac{1}{2}(\Delta t) u_{tt} - \frac{1}{3}\left(\Delta x_j - \Delta x_{j-1}\right) u_{xxx} \\ &\quad - \frac{1}{12}\left[(\Delta x_j)^2 + (\Delta x_{j-1})^2 - \Delta x_j \Delta x_{j-1}\right] u_{xxxx} + \cdots \end{aligned}$$

The result follows since $u_t = u_{xx}$.

(b) Suppose now that the boundary and initial conditions $u(0,t), u(1,t),$ and $u(x,0)$ are provided. Let $\Delta x = \max \Delta x_j$ and suppose the mesh is sufficiently regular such that $|\Delta x_j - \Delta x_{j-1}| \leq \alpha(\Delta x)^2$ for every $j = 1, 2, 3, \cdots, J-1$, where $\alpha > 0$ is constant.

Show that

$$|v_j^n - u(x_j, t_n)| \leq \left(\frac{1}{2}\Delta t\ M_{tt} + (\Delta x)^2 \left\{\frac{1}{3}\alpha M_{xxx} + \frac{1}{12}[1 + \alpha \Delta x] M_{xxxx}\right\}\right) t_f$$

provided that the stability condition

$$\Delta t \le \frac{1}{2}\Delta x_{j-1}\Delta x_j, \quad j = 1, 2, \cdots, J-1,$$

is satisfied.

Solution:

Let $e_j^n = v_j^n - u(x_j, t_n)$ be the error at grid point (x_j, t_n). Then by the definition of truncation error it follows that

$$\begin{aligned}
e_j^{n+1} &= e_j^n + \frac{2\Delta t}{\Delta x_j + \Delta x_{j-1}}\left(\frac{e_{j+1}^n - e_j^n}{\Delta x_j} - \frac{e_j^n - e_{j-1}^n}{\Delta x_{j-1}}\right) - T_j^n \Delta t \\
&= e_j^n - \frac{2\Delta t}{\Delta x_j + \Delta x_{j-1}}\left(\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j-1}}\right) e_j^n \\
&\quad + \frac{2\Delta t}{\Delta x_j + \Delta x_{j-1}}\frac{e_{j+1}^n}{\Delta x_j} + \frac{2\Delta t}{\Delta x_j + \Delta x_{j-1}}\frac{e_{j-1}^n}{\Delta x_{j-1}} - T_j^n \Delta t \\
&= \left(1 - \frac{2\Delta t}{\Delta x_j \Delta x_{j-1}}\right) e_j^n + \frac{2\Delta t}{\Delta x_j + \Delta x_{j-1}}\frac{e_{j+1}^n}{\Delta x_j} \\
&\quad + \frac{2\Delta t}{\Delta x_j + \Delta x_{j-1}}\frac{e_{j-1}^n}{\Delta x_{j-1}} - T_j^n \Delta t
\end{aligned}$$

Define $E^n := \max_j |e_j^n|$, $T^n := \max |T_j^n|$. To make all the coefficients positive, we require that $2\Delta t \le \Delta x_j \Delta_{j-1}$ then replacing all the e_ℓ^n for $\ell = j-1, j, j+1$ by the maximum value E^n and T_j^n by T^n

$$\begin{aligned}
E^{n+1} &\le \left(1 - \frac{2\Delta t}{\Delta x_j \Delta_{j-1}}\right) E^n + \frac{2\Delta t}{\Delta x_j \Delta x_{j-1}} E^n + T^n \Delta t \\
&= E^n + T^n \Delta t
\end{aligned}$$

Since $E^0 = 0$, it follows that $E^n \le \Delta t(T^0 + T^1 + \cdots + T^{n-1})$. Take $T = \max_n T^n$. Then

$$E^n \le n\Delta t T \le t_F T$$

where t_F is the final time. It remains to estimate T. Let $M_{tt}, M_{xxx}, M_{xxxx}$ be the maxima of the corresponding derivatives on the space-time domain. We note that

$$|\Delta x_j - \Delta x_{j-1}| \le \alpha(\Delta x)^2$$

and

$$(\Delta x_j)^2 + (\Delta x_{j-1})^2 - \Delta x_j \Delta x_{j-1} = \Delta x_j(\Delta x_j - \Delta x_{j-1}) + (\Delta x_{j-1})^2$$

The right hand side can be replaced by the upper bound

$$\Delta x \cdot \alpha(\Delta x)^2 + (\Delta x)^2 = (\Delta x)^2 \left(1 + \alpha \Delta x\right).$$

3. (a) Show that the leading terms in the truncation error of the Peaceman-Rachford ADI method for the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy}$$

are

$$\begin{aligned} T^{n+1/2} &= (\Delta t)^2 \left[\frac{1}{24} u_{ttt} - \frac{1}{8} (u_{xxtt} + u_{yytt}) + \frac{1}{4} u_{xxyyt} \right] \\ & -\frac{1}{12} \left[(\Delta x)^2 u_{xxxx} + (\Delta y)^2 y_{yyyy} \right]. \end{aligned}$$

Solution:

The Peaceman-Rachford scheme can be expressed in the form

$$\left(1 - \frac{1}{2}\mu_x \delta_x^2\right)\left(1 - \frac{1}{2}\mu_y \delta_y^2\right) u^{n+1} = \left(1 + \frac{1}{2}\mu_x \delta_x^2\right)\left(1 + \frac{1}{2}\mu_y \delta_y^2\right) u^n$$

where

$$\mu_x = \frac{\Delta t}{(\Delta x)^2}, \quad \mu_y = \frac{\Delta t}{(\Delta y)^2}$$

and

$$\delta_x^2 u^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$$

In expanded form:

$$\left(1 - \frac{1}{2}\mu_x \delta_x^2 - \frac{1}{2}\mu_y \delta_y^2 + \frac{1}{4}\mu_x \mu_y \delta_x^2 \delta_y^2\right) u^{n+1} = \left(1 + \frac{1}{2}\mu_x \delta_x^2 + \frac{1}{2}\mu_y \delta_y^2 + \frac{1}{4}\mu_x \mu_y \delta_x^2 \delta_y^2\right) u^n$$

Taking all the terms to the right hand side, and grouping:

$$(u^{n+1} - u^n) - \frac{1}{2}\mu_x \delta_x^2 (u^{n+1} + u_n) - \frac{1}{2}\mu_y \delta_y^2 (u^{n+1} + u_n) + \frac{1}{4}\mu_x \mu_y \delta_x^2 \delta_y^2 (u^{n+1} - u^n) = 0,$$

which is equivalent to

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{1}{2}\frac{\delta_x^2}{(\Delta x)^2}(u^{n+1} + u_n) - \frac{1}{2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} + u_n) + \frac{1}{4}(\Delta t)\frac{\delta_x^2}{(\Delta x)^2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} - u^n) = 0,$$

Using Taylor series to expand around the point $(x_j, t_{n+1/2})$ we obtain

$$u^{n+1} = u + \frac{1}{2}(\Delta t) u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} + \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \cdots$$

$$u^n = u - \frac{1}{2}(\Delta t) u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} - \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \cdots.$$

It follows that

$$\frac{u^{n+1} - u^n}{\Delta t} = u_t + \frac{1}{24}(\Delta t)^2 u_{ttt} + \cdots$$

and

$$u^{n+1} + u^n = 2u + \frac{1}{4}(\Delta t)^2 u_{tt} + \cdots$$

Recall Equation 2.30 pg 14 of Morton

$$\frac{\delta_x^2 u}{(\Delta x)^2} = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots$$

From the ADI scheme

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{1}{2}\frac{\delta_x^2}{(\Delta x)^2}(u^{n+1} + u_n) - \frac{1}{2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} + u_n) + \frac{1}{4}(\Delta t)\frac{\delta_x^2}{(\Delta x)^2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} - u^n) = 0,$$

we see that

$$\frac{1}{2}\frac{\delta_x^2}{(\Delta x)^2}(u^{n+1} + u^n) = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \frac{1}{8}(\Delta t)^2 u_{xxtt} + \cdots$$

The Taylor expansion of the y-term is similar. The expansion of the mixed term is

$$\frac{1}{4}(\Delta t)\frac{\delta_x^2}{(\Delta x)^2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} - u^n) = \frac{1}{4}(\Delta t)^2 u_{xxyyt} + \frac{1}{96}(\Delta t)^4 u_{xxyyttt} + \cdots$$

Combining these terms, we get

$$\begin{aligned} T_j^n &= (u_t - u_{xx} - u_{yy}) + \frac{1}{24}(\Delta t)^2 u_{ttt} \\ &\quad - \frac{1}{12}(\Delta x)^2 u_{xxxx} - \frac{1}{8}(\Delta t)^2 u_{xxtt} \\ &\quad - \frac{1}{12}(\Delta y)^2 u_{yyyy} - \frac{1}{8}(\Delta t)^2 u_{yytt} + \frac{1}{4}(\Delta t)^2 u_{xxyyt} + \cdots \end{aligned}$$

Since $u_t = u_{xx} + u_{yy}$, the first term vanishes.

(b) Show that the Douglas-Rachford scheme

$$\begin{aligned} \left(1 - \mu_x \delta_x^2\right) v^{n+1*} &= \left(1 + \mu_y \delta_y^2 + \mu_z \delta_z^2\right) v^n \\ \left(1 - \mu_y \delta_y^2\right) v^{n+1**} &= v^{n+1*} - \mu_y \delta_y^2 v^n \\ \left(1 - \mu_z \delta_z^2\right) v^{n+1} &= v^{n+1**} - \mu_z \delta_z^2 v^n \end{aligned}$$

for the three-dimensional heat equation

$$u_t = u_{xx} + u_{yy} + u_{zz}$$

is unconditionally stable when applied to a rectilinear box.

Solution:

Consider a solution of the form

$$v_{j,\ell,k}^{n} = \lambda^n e^{i(\xi_x j\Delta x + \xi_y \ell \Delta y + \xi_x k \Delta z)}.$$

Set

$$v_{j,\ell,k}^{n+1\star} = \gamma\lambda^n e^{i(\xi_x j\Delta x + \xi_y \ell \Delta y + \xi_x k \Delta z)},$$

and

$$v_{j,\ell,k}^{n+1\star\star} = \beta\lambda^n e^{i(\xi_x j\Delta x + \xi_y \ell \Delta y + \xi_x k \Delta z)}.$$

Substituting these solutions into the difference scheme yields,

$$\begin{aligned}
\gamma\left(1+4\mu_x\sin^2(\xi_x\Delta x)\right) &= \left(1-4\mu_y\sin^2(\xi_y\Delta y)-4\mu_z\sin^2(\xi_z\Delta z)\right) && (1)\\
\beta\left(1+4\mu_y\sin^2(\xi_y\Delta y)\right) &= \gamma+4\mu_y\sin^2(\xi_y\Delta y) && (2)\\
\lambda\left(1+4\mu_z\sin^2(\xi_z\Delta z)\right) &= \beta+4\mu_z\sin^2(\xi_z\Delta z) && (3)
\end{aligned}$$

Multiplying the second equation by $\left(1+4\mu_x\sin^2(\xi_x\Delta x)\right)$, we can use the first equation to eliminate γ

$$\begin{aligned}
\beta\left(1+4\mu_x\sin^2(\xi_x\Delta x)\right)\left(1+4\mu_y\sin^2(\xi_y\Delta y)\right) &= \left(1-4\mu_y\sin^2(\xi_y\Delta y)-4\mu_z\sin^2(\xi_z\Delta z)\right)\\
&\quad +4\mu_y\sin^2(\xi_y\Delta y)\left(1+4\mu_x\sin^2(\xi_x\Delta x)\right)\\
&= 1-4\mu_z\sin^2(\xi_z\Delta z)+16\mu_x\mu_y\sin^2(\xi_x\Delta x)\sin^2(\xi_y\Delta y)
\end{aligned}$$

Now multiply Equation 3 by

$$\left(1+4\mu_x\sin^2(\xi_x\Delta x)\right)\left(1+4\mu_y\sin^2(\xi_y\Delta y)\right)$$

to eliminate β. We get, finally

$$\lambda = \frac{1+A_{xy}+A_{xz}+A_{yz}+B_{xyz}}{\left(1+4\mu_x\sin^2(\xi_x\Delta x)\right)\left(1+4\mu_y\sin^2(\xi_y\Delta y)\right)\left(1+4\mu_z\sin^2(\xi_z\Delta z)\right)} \le 1$$

where

$$\begin{aligned}
A_{xy} &= 16\mu_x\mu_y\sin^2(\xi_x\Delta x)\sin^2(\xi_y\Delta y)\\
A_{xz} &= 16\mu_x\mu_z\sin^2(\xi_x\Delta x)\sin^2(\xi_z\Delta z)\\
A_{yz} &= 16\mu_y\mu_z\sin^2(\xi_y\Delta y)\sin^2(\xi_z\Delta z)\\
B_{xyz} &= 64\mu_x\mu_y\mu_z\sin^2(\xi_x\Delta x)\sin^2(\xi_y\Delta y)\sin^2(\xi_z\Delta z).
\end{aligned}$$