Unconditional Stability of the ADI scheme

MTSCS546

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Claim:
The 2D Douglas-Rachford scheme for the heat equation $u_t = \Delta u$ is unconditionally stable.

Proof:
Recall that the Douglas-Rachford scheme for the heat equation (using the notation of Morton) takes the form

$$
\begin{aligned}
(1-\mu_x\delta_x^2)v^{n+1\star} &= (1+\mu_y\delta_y^2)v^n && \text{(1a)}\\
(1-\mu_y\delta_y^2)v^{n+1} &= v^{n+1\star} - \mu_y\delta_y^2 v^n && \text{(1b)}
\end{aligned}
$$

Suppose the solution of the scheme is in the form

$$v_{j,\ell}^n = \lambda^n e^{i\xi j\Delta x} e^{i\eta\ell\Delta y}.$$

$$v_{j,\ell}^{n+1\star} = \beta\lambda^n e^{i\xi j\Delta x} e^{i\eta\ell\Delta y}$$

Applying Von Neumann stability analysis to (1), we get

$$
\begin{aligned}
\left(1+4\mu_x\sin^2\frac{\theta_x}{2}\right)\beta &= \left(1-4\mu_y\sin^2\frac{\theta_y}{2}\right) && \text{(2a)}\\
\left(1+4\mu_y\sin^2\frac{\theta_y}{2}\right)\lambda &= \beta + 4\mu_y\sin^2\frac{\theta_y}{2} && \text{(2b)}
\end{aligned}
$$

Multiply the second equation in (2) by $1+4\mu_x\sin^2\frac{\theta_x}{2}$ and eliminate the top equation

$$
\begin{aligned}
\left(1+4\mu_x\sin^2\frac{\theta_x}{2}\right)\left(1+4\mu_y\sin^2\frac{\theta_y}{2}\right)\lambda &= \left(1-4\mu_y\sin^2\frac{\theta_y}{2}\right) + 4\mu_y\sin^2\frac{\theta_y}{2}\left(1+4\mu_x\sin^2\frac{\theta_x}{2}\right)\\
&= 1+16\mu_x\mu_y\sin^2\frac{\theta_x}{2}\sin^2\frac{\theta_y}{2}
\end{aligned}
$$

Hence,

$$\lambda = \frac{1+16\mu_x\mu_y\sin^2\frac{\theta_x}{2}\sin^2\frac{\theta_y}{2}}{\left(1+4\mu_x\sin^2\frac{\theta_x}{2}\right)\left(1+4\mu_y\sin^2\frac{\theta_y}{2}\right)} \leq 1$$