Truncation Error of the ADI scheme

MTSCS546

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Show that the leading terms of the truncation error of the 2D Peaceman-Rachford ADI scheme for the heat equation $u_t = \Delta u$ are given by

$$\begin{aligned} T^{n+1/2} &= (\Delta t)^2 \left[\frac{1}{24} u_{ttt} - \frac{1}{8}(u_{xxtt} + u_{yytt}) + \frac{1}{4} u_{xxyyt} \right] \\ & -\frac{1}{12} \left[(\Delta x)^2 u_{xxxx} + (\Delta y)^2 u_{yyyy} \right] \end{aligned}$$

Proof:
The Peaceman-Rachford scheme can be expressed in the form

$$\left(1 - \frac{1}{2}\mu_x \delta_x^2\right)\left(1 - \frac{1}{2}\mu_y \delta_y^2\right) u^{n+1} = \left(1 + \frac{1}{2}\mu_x \delta_x^2\right)\left(1 + \frac{1}{2}\mu_y \delta_y^2\right) u^n$$

where

$$\mu_x = \frac{\Delta t}{(\Delta x)^2}, \quad \mu_y = \frac{\Delta t}{(\Delta y)^2}$$

and

$$\delta_x^2 u^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$$

In expanded form:

$$\left(1 - \frac{1}{2}\mu_x \delta_x^2 - \frac{1}{2}\mu_y \delta_y^2 + \frac{1}{4}\mu_x \mu_y \delta_x^2 \delta_y^2\right) u^{n+1} = \left(1 + \frac{1}{2}\mu_x \delta_x^2 + \frac{1}{2}\mu_y \delta_y^2 + \frac{1}{4}\mu_x \mu_y \delta_x^2 \delta_y^2\right) u^n$$

Taking all the terms to the right hand side, and grouping:

$$(u^{n+1} - u^n) - \frac{1}{2}\mu_x \delta_x^2 (u^{n+1} + u_n) - \frac{1}{2}\mu_y \delta_y^2 (u^{n+1} + u_n) + \frac{1}{4}\mu_x \mu_y \delta_x^2 \delta_y^2 (u^{n+1} - u^n) = 0,$$

which is equivalent to

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{1}{2}\frac{\delta_x^2}{(\Delta x)^2}(u^{n+1} + u_n) - \frac{1}{2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} + u_n) + \frac{1}{4}(\Delta t)\frac{\delta_x^2}{(\Delta x)^2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} - u^n) = 0,$$

Using Taylor series to expand around the point $(x_j, t_{n+1/2})$ we obtain

$$u^{n+1} = u + \frac{1}{2}(\Delta t) u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} + \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \cdots$$

$$u^n = u - \frac{1}{2}(\Delta t)u_t + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 u_{tt} - \frac{1}{6}\left(\frac{1}{2}\Delta t\right)^3 u_{ttt} + \cdots.$$

It follows that

$$\frac{u^{n+1} - u^n}{\Delta t} = u_t + \frac{1}{24}(\Delta t)^2 u_{ttt} + \cdots$$

and

$$u^{n+1} + u^n = 2u + \frac{1}{4}(\Delta t)^2 u_{tt} + \cdots$$

Recall that

$$\frac{\delta_x^2 u}{(\Delta x)^2} = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots$$

From the ADI scheme

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{1}{2}\frac{\delta_x^2}{(\Delta x)^2}(u^{n+1} + u_n) - \frac{1}{2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} + u_n) + \frac{1}{4}(\Delta t)\frac{\delta_x^2}{(\Delta x)^2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} - u^n) = 0,$$

we see that

$$\frac{1}{2}\frac{\delta_x^2}{(\Delta x)^2}(u^{n+1} + u^n) = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \frac{1}{8}(\Delta t)^2 u_{xxtt} + \cdots$$

The Taylor expansion of the y-term is similar. The expansion of the mixed term is

$$\frac{1}{4}(\Delta t)\frac{\delta_x^2}{(\Delta x)^2}\frac{\delta_y^2}{(\Delta y)^2}(u^{n+1} - u^n) = \frac{1}{4}(\Delta t)^2 u_{xxyyt} + \frac{1}{96}(\Delta t)^4 u_{xxyyttt} + \cdots$$

The result is obtained by grouping these expansions.