HMTHCS 212 Assignment 2 Name ______________________________

This assignment is due on **Wednesday July 13, 2022** by 6pm.

1. (30 points)

   (a) Use Gaussian elimination solve the following system of equations, if possible, and determine if row interchanges are necessary.

$$\begin{aligned} x_1 + x_2 + x_4 &= 2 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \end{aligned}$$

Solution: In matrix form:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 4 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right) \xrightarrow[R4-3R1]{\substack{R2-2R1 \\ R3+R1}} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right) \xrightarrow[R4+4R2]{R3+3R2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & \boxed{0} & -3 & -3 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right)$$

At this stage naive Gaussian elimination breaks down because of the boxed zero pivot. Let us interchange row 3 and row 4, and continue with Gaussian elimination.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & \boxed{0} & -3 & -3 \end{array}\right)$$

The matrix is now upper triangular. Solving by back-substitution

$$-3x_4 = -3 \Longrightarrow \boxed{x_4 = 1}$$

$$3x_3 + 3 \cdot 1 = 3 \Longrightarrow 3x_3 = 0 \Longrightarrow \boxed{x_3 = 0}$$

$$-x_2 - 1 = -3 \Longrightarrow -x_2 = -2 \Longrightarrow \boxed{x_2 = 2}$$

$$x_1 + 2 + 1 = 2 \Longrightarrow \boxed{x_1 = -1}$$

(b) Use Gaussian elimination with backward substitution to solve the following linear system. Do not re-order the equations.

$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \end{aligned}$$

Solution:

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right) \xrightarrow[R3-\frac{1}{4}R1]{R2-\frac{1}{2}R1} \left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & 7/2 & -2 & -19/2 \\ 0 & 3/4 & -7/2 & -45/4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & 7/2 & -2 & -19/2 \\ 0 & 3/4 & -7/2 & -45/4 \end{array}\right) \xrightarrow{R3-\frac{3}{14}R2} \left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & 7/2 & -2 & -19/2 \\ 0 & 0 & -43/14 & -129/14 \end{array}\right)$$

Now using back-substitution, we have

$$-\frac{43}{14}x_3 = -\frac{129}{14} \implies \boxed{x_3 = 3}$$
$$\frac{7}{2}x_2 - 2\cdot 3 = -\frac{19}{2} \implies \boxed{x_2 = -1}$$
$$4x_1 - 1 + 2\cdot 3 = 9 \implies \boxed{x_1 = 1}$$

(c) Show that the LU factorization of the matrix A from part (b) is $A = LU$, where:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{3}{14} & 1 \end{pmatrix}, \text{ and } U = \begin{pmatrix} 4 & 1 & 2 \\ 0 & \frac{7}{2} & -2 \\ 0 & 0 & -\frac{43}{14} \end{pmatrix}$$

Solution: The matrix U is simply the final upper triangular matrix obtained after Gaussian elimination so that

$$U = \begin{pmatrix} 4 & 1 & 2 \\ 0 & \frac{7}{2} & -2 \\ 0 & 0 & -\frac{43}{14} \end{pmatrix}$$

The lower triangular matrix is obtained by placing 1's along the diagonal, and the multipliers from Gaussian elimination in the lower block, followed by zeros in the upper block. For example, we subtracted $\frac{1}{2}R1$ from $R2$ at the beginning. So the corresponding entry in L is $\frac{1}{2}$ and so forth.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 3/14 & 1 \end{pmatrix}$$

(d) Use the LU factorization from part (c) to solve the linear system in part (b).

Solution: Since $A = LU$, solving $Ax = b$ is similar to solving $LUx = b$, which is identical to solving $Ly = b$ for $y = (y_1, y_2, y_3)$ followed by solving $Ux = y$ for $x = (x_1, x_2, x_3)$.

Let us solve $Ly = b$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 3/14 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \\ -9 \end{pmatrix}.$$

By forward substitution,

$$y_1 = 9$$

$$\frac{1}{2} \cdot 9 + y_2 = -5 \Longrightarrow y_2 = -\frac{19}{2}$$

$$\frac{1}{4} \cdot 9 + \frac{3}{14} \cdot -\frac{19}{2} + y_3 = -9 \Longrightarrow y_3 = -\frac{129}{14}.$$

Then we solve $Ux = y$ for x.

$$\begin{pmatrix} 4 & 1 & 2 \\ 0 & \frac{7}{2} & -2 \\ 0 & 0 & -\frac{43}{14} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ -19/2 \\ -129/14 \end{pmatrix}$$

which we can solve by back-substitution as in the final step of Gaussian elimination to get

$$\boxed{x_1 = 1, \quad x_2 = -1, \quad x_3 = 0, \quad x_4 = 3}$$

(e) What is a diagonally dominant matrix? Show that the matrix A from part (b) is diagonally dominant.

Solution: A matrix $A = (a_{ij})_{1 \le i,j \le n}$ is called **diagonally dominant** if the magnitude of the diagonal entry is not less than the sum of the magnitudes of all other entries in the row containing the diagonal. That is

$$|a_{ii}| \ge \sum_{\substack{i \ne j \\ j=1}}^{n} |a_{ij}|$$

It is **strictly diagonally dominant** if

$$|a_{ii}| > \sum_{\substack{i \ne j \\ j=1}}^{n} |a_{ij}|$$

(f) Perform 3 steps of the Jacobi iteration to approximate the solution of the linear system from part (b).

Solution: Since the matrix is **strictly diagonally dominant**, the Jacobi iteration converges to the solution. The Jacobi method is the iteration

$$\begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \\ x_3^{(n+1)} \end{pmatrix} = \begin{pmatrix} [9 - x_2^{(n)} - 2x_3^{(n)}]/4 \\ [-5 - 2x_1^{(n)} + x_3^{(n)}]/4 \\ [9 + x_1^{(n)} + x_2^{(n)}]/3 \end{pmatrix}$$

Starting with an initial guess, say

$$\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} &= \begin{pmatrix} 9/4 \\ -5/4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.25 \\ -1.25 \\ 3 \end{pmatrix} \\
\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{pmatrix} &= \begin{pmatrix} [9 - x_2^{(1)} - 2x_3^{(1)}]/4 \\ [-5 - 2x_1^{(1)} + x_3^{(1)}]/4 \\ [9 + x_1^{(1)} + x_2^{(1)}]/3 \end{pmatrix} = \begin{pmatrix} 17/16 \\ -13/8 \\ 10/3 \end{pmatrix} \approx \begin{pmatrix} 1.0625 \\ -1.625 \\ 3.333 \end{pmatrix} \\
\begin{pmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{pmatrix} &= \begin{pmatrix} [9 - x_2^{(2)} - 2x_3^{(2)}]/4 \\ [-5 - 2x_1^{(2)} + x_3^{(2)}]/4 \\ [9 + x_1^{(2)} + x_2^{(2)}]/3 \end{pmatrix} = \begin{pmatrix} 95/96 \\ -91/96 \\ 45/16 \end{pmatrix} \approx \begin{pmatrix} 0.98958 \\ -0.94792 \\ 2.8125 \end{pmatrix}
\end{aligned}$$

(g) Perform 3 steps of the Gauss-Seidel iteration to approximate the solution of the linear system from part (b).

Solution: The Gauss-Seidel iteration is as follows

$$\begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \\ x_3^{(n+1)} \end{pmatrix} = \begin{pmatrix} [9 - x_2^{(n)} - 2x_3^{(n)}]/4 \\ [-5 - 2x_1^{(n+1)} + x_3^{(n)}]/4 \\ [9 + x_1^{(n+1)} + x_2^{(n+1)}]/3 \end{pmatrix}$$

Observe the difference (in red) between Jacobi and Gauss-Seidel. Starting with an initial guess, say

$$\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} [9 - x_2^{(0)} - 2x_3^{(0)}]/4 \\ [-5 - 2x_1^{(1)} + x_3^{(n)}]/4 \\ [9 + x_1^{(1)} + x_2^{(1)}]/3 \end{pmatrix} = \begin{pmatrix} 9/4 \\ -19/8 \\ 71/24 \end{pmatrix} = \begin{pmatrix} 2.25 \\ -2.375 \\ 2.9583 \end{pmatrix}$$

$$\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{pmatrix} = \begin{pmatrix} [9 - x_2^{(1)} - 2x_3^{(1)}]/4 \\ [-5 - 2x_1^{(2)} + x_3^{(1)}]/4 \\ [9 + x_1^{(2)} + x_2^{(2)}]/3 \end{pmatrix} = \begin{pmatrix} 131/96 \\ -229/192 \\ 587/192 \end{pmatrix} \approx \begin{pmatrix} 1.36458 \\ -1.19271 \\ 3.05729 \end{pmatrix}$$

$$\begin{pmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{pmatrix} = \begin{pmatrix} [9 - x_2^{(2)} - 2x_3^{(2)}]/4 \\ [-5 - 2x_1^{(3)} + x_3^{(2)}]/4 \\ [9 + x_1^{(3)} + x_2^{(3)}]/3 \end{pmatrix} = \begin{pmatrix} 261/256 \\ -1529/1536 \\ 749/249 \end{pmatrix} \approx \begin{pmatrix} 1.01953 \\ -0.99544 \\ 3.00803 \end{pmatrix}$$

2. (20 points) Consider the following table of data. Find a polynomial of degree 3 that

x	-1	0	1	2
y	3	5	5	27

interpolates this data using the following methods. Show that the polynomial is the same in all three cases.

(a) monomial interpolation

Solution: Form a system of equations by plugging in the points into a polynomial of the form

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

The polynomial is of degree 3 because there are 4 points.

$$p(-1) = -a_3 + a_2 - a_1 + a_0 = 3$$

$$p(0) = a_0 = 5$$

$$p(1) = a_3 + a_2 + a_1 + a_0 = 5$$

$$p(2) = 8a_3 + 4a_2 + 2a_1 + a_0 = 27$$

Writing in matrix format we get the system

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 5 \\ 27 \end{pmatrix}$$

which we proceed to solve by Gaussian elimination. First, interchange rows two and four.

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 3 \\ 8 & 4 & 2 & 1 & 27 \\ 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right) \xrightarrow[R3+R1]{R2+8R1} \left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 3 \\ 0 & 12 & -6 & 9 & 51 \\ 0 & 2 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 3 \\ 0 & 12 & -6 & 9 & 51 \\ 0 & 2 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right) \xrightarrow{R3-\frac{1}{6}R2} \left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 3 \\ 0 & 12 & -6 & 9 & 51 \\ 0 & 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

Using back-substitution

$$\boxed{a_0 = 5}$$

$$a_1 + \frac{1}{2}(5) = -\frac{1}{2} \Longrightarrow \boxed{a_1 = -3}$$

$$12a_2 + 18 + 45 = 51 \Longrightarrow \boxed{a_2 = -1}$$

$$-a_3 - 1 + 3 + 5 = 3 \Longrightarrow \boxed{a_3 = 4}$$

Therefore

$$\boxed{p(x) = 4x^3 - x^2 - 3x + 5}.$$

(b) Lagrange's interpolation

Solution: The Lagrange interpolant in this case is

$$\begin{aligned} p(x) &= 3\frac{x(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} + 5\frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)} \\ &\quad +5\frac{x(x+1)(x-2)}{(1+1)(1-0)(1-2)} + 27\frac{x(x+1)(x-1)}{(2+1)(2-0)(2-1)} \\ &= -\frac{1}{2}x(x-1)(x-2) + \frac{5}{2}(x+1)(x-1)(x-2) \\ &\quad -\frac{5}{2}x(x+1)(x-2) + \frac{9}{2}x(x+1)(x-1) \end{aligned}$$

Expanding this, we get

$$\begin{aligned} p(x) &= x^3\left[-\frac{1}{2} + \frac{5}{2} - \frac{5}{2} + \frac{9}{2}\right] + x^2\left[\frac{3}{2} - 5 + \frac{5}{2}\right] \\ &\quad +x\left[-1 - \frac{5}{2} + 5 - \frac{9}{2}\right] + 5 \\ &= 4x^3 - x^2 - 3x + 5. \end{aligned}$$

-1	$\boxed{3}$			
		$\boxed{2}$		
0	5		$\boxed{-1}$	
		0		$\boxed{4}$
1	5		11	
		22		
2	27			

(c) Newton's interpolation (using a divided differences table)

Solution: Hence

$$p(x) = 3 + 2(x+1) - x(x+1) + 4x(x+1)(x-1)$$

Rearranging this, we get

$$\boxed{p(x) = 4x^3 - x^2 - 3x + 5}$$

3. (10 points) A quadratic polynomial $p(x)$ is used to approximate the function $f(x) = e^x$ on the interval $[-1, 1]$. The interpolating polynomial passes through the points $x = -1, 0, 1$

(a) Find the interpolating polynomial $p(x)$.

Solution: We can use any of the three methods to figure out the interpolating polynomial. For example, using Lagrange interpolation:

$$p(x) = e^{-1}\frac{x(x-1)}{(-1)(-2)} + e^0\frac{(x+1)(x-1)}{(1)(-1)} + e^1\frac{x(x+1)}{(2)(1)}$$

or

$$\boxed{p(x) = \frac{1}{2e}x(x-1) - (x+1)(x-1) + \frac{e}{2}x(x+1)}.$$

(b) Write down the expression for the interpolation error $E(x) = |e^x - p(x)|$

Solution: The interpolation error when approximating $f(x)$ by an interpolating polynomial $p(x)$ at points $(x_0, f(x_0)), \cdots (x_n, f(x_n))$ is given by

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\cdots(x - x_n)$$

for some $\xi \in (x_0, x_n)$ In this case $f(x) = e^x$, $x_0 = -1, x_1 = 0, x_2 = 1$. Since $n = 2$, we require $f^{(3)}(x) = e^x$.

$$|e^x - p(x)| = \frac{e^c}{6}x(x+1)(x-1)$$

for some $c \in (-1, 1)$.

(c) Hence, find the maximum possible value of the interpolation error $E(x)$ on the interval $[-1,1]$.

Solution: We want to compute

$$\max_{x\in[-1,1]} |e^x - p(x)| = \max_{x\in[-1,1]} \frac{|e^{c(x)}|}{6} x(x+1)(x-1)$$

Th function e^x is increasing on $[-1,1]$ so the maximum of $e^{c(x)}$ is $e^1 = e$. It remains to compute the maximum/minimum of $\phi(x) = x(x+1)(x-1) = x^3 - x$ on the interval $[-1,1]$. Since $\phi(-1) = \phi(1) = 0$ at the end-points, the maximum or minumum must occur at a critical point.

$$\phi'(x) = 3x^2 - 1 = 0$$

has solutions at $x = \pm\frac{1}{\sqrt{3}}$. Evaluating into $\phi(x)$ we get

$$\phi(1/\sqrt{3}) = -\frac{2}{3\sqrt{3}}$$

and

$$\phi(-1/\sqrt{3}) = \frac{2}{3\sqrt{3}}.$$

Hence

$$\max_{x\in[-1,1]} |e^x - p(x)| \leq \frac{e}{6} \cdot \frac{2}{3\sqrt{3}} = 0.17438$$