HMTHCS 212 Assignment 1 Name ____________________

This assignment is due on **Wednesday July 6, 2022** by 6pm.

1. (20 points)

   (a) Find a way to compute the following function to avoid loss of significant figures

   $$f(x) = \frac{\sin(x)}{x - \sqrt{x^2 - 1}}$$

   (b) For any $x_0 > -1$, the sequence defined recursively by

   $$x_{n+1} = 2^{n+1}\left(\sqrt{1 + 2^{-n}x_n} - 1\right), \quad n \geq 0$$

   converges to $\ln(x_0 + 1)$. Arrange this formula in a way that avoids loss of significance.

   (c) Calculate both roots of

   $$3x^2 - 9^{14}x + 100 = 0$$

   with 3 digit accuracy.

   (d) Find a good way of computing

   $$f(x) = \frac{e^{2x} - 1}{2x}$$

   for x near zero.

2. (20 points)

   (a) By forming a suitable function $f(x)$, show that the graphs of $u(x) = \frac{x}{2}$ and $v(x) = \tan^{-1}(x)$ intersect at three points: $x = \alpha$ in the interval $[2, 2.5]$, $x = 0$ and $x = -\alpha$.

   (b) Perform 6 steps of the bisection method to estimate α.

   (c) Find an interval $I \subset \mathbb{R}$ so that the fixed-point iteration $x_{n+1} = 2\tan^{-1}(x_n)$ starting from any $x_0 \in I$ is guaranteed to converge to α

   (d) Perform 6 steps of the fixed-point iteration to approximate α starting with $x_0 = 2$.

   (e) Explain why the fixed-point iteration $x_{n+1} = 2\tan^{-1}(x_n)$ is not guaranteed to converge to the root at $x = 0$.

   (f) For what starting values x_0 does Newton's method fail to find the roots of $f(x)$?

3. (20 points)

   (a) Verify that when Newton's method is used to compute $\sqrt[3]{a}$ (by solving the equation $x^3 = a$), the sequence of iterates is defined by

   $$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right).$$

(b) Perform three iterations of the scheme in part (a) for computing $\sqrt[3]{5}$, starting with $x_0 = 2$, and for the bisection method starting with the interval $[1, 2]$. How many iterations of the bisection method are required in order to obtain 10^{-6} accuracy?

(c) Suppose $xe^x = 3$. By drawing suitable graphs, find a first approximation x_0 of the root of $xe^x = 3$ as the intersection of two graphs.

(d) Hence use Newton's method to find the root of $xe^x = 3$ correct to 3 decimal places.

(e) Explain what would happen if we were to choose $x_0 = -1$ as the first approximation in Newton's method for the root in (c).