HMTHCS 212 Assignment 3 Name ____________________

This assignment is due by **Monday July 18, 2022** by 6pm.

1. (10 points)

   (a) Find the values of the coefficients α, β and order of convergence p such that $f'(x)$ is best approximated by the formula

   $$f'(x) = \frac{\alpha f(x+h) + \beta f(x-2h)}{h} + \mathcal{O}(h^p)$$

   (b) Using the formula derived from part (a), find an approximation of the derivative $f'(a)$ for the function $f(x) = \ln(x)$ at $a = 1$ for the following values of h (correct to 6 decimal places).

   (i) $h = 0.1$

   (ii) $h = 0.05$

   (iii) $h = 0.025$

   (c) Compute the exact value of the derivative $f'(1)$ and then find the absolute errors of the approximations for each of the three values of h in part (b) (correct to 6 decimal places).

   (d) Let E_h be the absolute error for the approximation of the derivative $f'(1)$ from part (c) for a value of h. By computing $E_{0.1}/E_{0.05}$ and $E_{0.05}/E_{0.025}$, comment on the order of convergence.

2. (10 points)

   (a) Find the values of the weights a_0, a_1 and a_2 such that the quadrature formula

   $$\int_0^1 f(x)\, dx \approx a_0 f(0) + a_1 f(0.25) + a_2 f(1)$$

   has the highest possible degree of precision.

   (b) Use the quadrature formula from part (b) to approximate the value of the integral

   $$\int_0^1 \frac{1}{1+x^2}\, dx.$$

   Compute the exact value of this integral and find the absolute error of the approximation from part (a).

   (c) Show that Gaussian quadrature with 3 nodes and weights

   $$\int_{-1}^1 f(x)\, dx \approx \frac{5}{9} f\left(-\sqrt{3/5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{3/5}\right)$$

   has degree of precision equal to 5.

3. (10 points)

(a) The left end-point rule is defined as follows: for nodes x_0 and x_1, approximate the integral by the area of a rectangle whose height is evaluated at the left end-point of the interval. Using Taylor series centered at $x = x_0$, and expanding $f(x)$ up to the linear term only, show that the left end-point rule with remainder is given by

$$\int_{x_0}^{x_1} f(x)\,dx = hf(x_0) + \frac{h^2}{2}f'(c)$$

for some $c \in (x_0, x_1)$.

(b) Generalize the method developed in part (a) to show that the composite left-end point method with n sub-divisions is given by

$$\int_a^b f(x)\,dx = h\sum_{k=0}^{n-1} f(a+kh) + (b-a)\frac{h}{2}f'(c)$$

where $c \in (a, b)$.

(c) Find the minimum number of sub-divisions n such that the integral

$$\int_0^1 x^2\,dx$$

can be approximated by the left end-point rule up to an error of at most 0.001. Hence find the minimum value of h that corresponds to this n.

4. (10 points)

Consider the integral

$$I = \int_0^\pi \exp(\sin(x))\,dx \approx 6.208758035711110.$$

(a) Use the Trapezoidal rule with $n = 4$ sub-divisions to approximate I.

(b) Use Simpson's method with $n = 2 \cdot 2 = 4$ sub-divisions to approximate I.

(c) Use Gaussian quadrature with $n = 3$ nodes to approximate I.

(d) The graph of $f(x) = \exp(\sin(x))$ and its derivatives $f'(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$ and $f^{(4)}(x)$ on $[0, \pi]$ is depicted below.

Using information from this graph answer the following questions.

(i) Find the maximum value of h such that Simpson's rule applied to the integral I results in an error not greater than 10^{-3}.

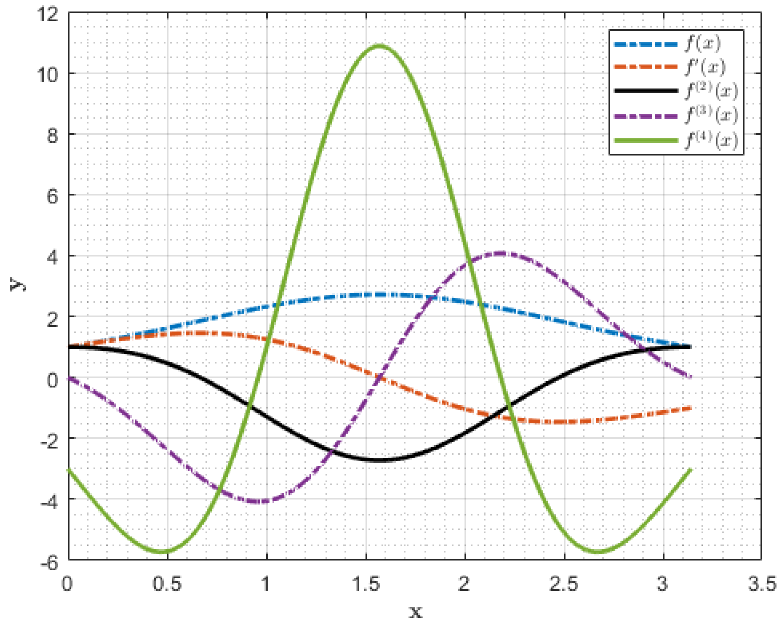
(ii) Find the number of sub-divisions n such that the Trapezoidal rule applied to I results in an error not greater than 10^{-3}.

5. (10 points)

Consider the initial value problem

$$y'(t) = y(t) - e^{-t}, \quad y(0) = 1, \quad t \in [0, 0.03]$$

The exact solution is $y(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$.



(a) Use Euler's method with step size $h = 0.01$ to approximate $y(0.03)$. Compute the absolute error of the approximation at $t = 0.03$.

(b) The modified Euler's method (Heun's method) is defined as

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right], \quad y_0 = y(0).$$

Use the modified Euler's method to approximate the value of $y(0.03)$ using a step size of $h = 0.01$. Compute the absolute error.

(c) Is the modified Euler's method an explicit or implicit method? Explain.

6. (10 points)

The Crank-Nicholson's scheme for approximating the initial value problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0$$

is defined as follows

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

where y_{n+1} is the unknown.

(a) By integrating the ODE on the interval $[t_n, t_{n+1}]$, and using the Trapezoidal rule to approximate the integral, derive the Crank-Nicholson scheme.

(b) Use the Crank-Nicholson scheme with $h = 0.01$ to approximate $y(0.03)$ for the initial value problem in 5(a). Compute the absolute error.

(c) Is the Crank-Nicholson scheme an explicit or implicit method? Explain.

(d) Use the fourth-order Runge-Kutta (RK-4) method to approximate $y(0.03)$ for the initial value problem in 5(a). Compute the absolute error.