HMTHCS 212 Assignment 1 Name ____________________

This assignment is due on **Wednesday July 6, 2022** by 6pm.

1. (20 points)

   (a) Find a way to compute the following function to avoid loss of significant figures

$$f(x) = \frac{\sin(x)}{x - \sqrt{x^2 - 1}}$$

Solution:

The evaluation of $f(x)$ may suffer from cancellation when x is large because in this case $x \approx \sqrt{x^2 - 1}$. The denominator approaches zero for large x. To avoid cancellation, re-write f using conjugate multiplication.

$$\begin{aligned} f(x) = \frac{\sin(x)}{x - \sqrt{x^2 - 1}} &= \frac{\sin(x)}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} \\ &= \frac{\sin(x) \cdot (x + \sqrt{x^2 - 1})}{x^2 - (x^2 - 1)} \\ &= \boxed{\sin(x) \cdot (x + \sqrt{x^2 - 1})} \\ &\approx 2x \sin(x), \text{ for large } x. \end{aligned}$$

   (b) For any $x_0 > -1$, the sequence defined recursively by

$$x_{n+1} = 2^{n+1} \left(\sqrt{1 + 2^{-n} x_n} - 1 \right), \quad n \geq 0$$

converges to $\ln(x_0 + 1)$. Arrange this formula in a way that avoids loss of significance.

Solution:

As $n \to \infty$, the term $\sqrt{1 + 2^{-n} x_n} \to 1$, since $2^{-n} \to 0$ as $n \to \infty$. Hence there is potential for loss of significance of $\sqrt{1 + 2^{-n} x_n} - 1$ when n is large. Using conjugate multiplication,

$$\begin{aligned} x_{n+1} &= 2^{n+1} \left(\sqrt{1 + 2^{-n} x_n} - 1 \right) \frac{\left(\sqrt{1 + 2^{-n} x_n} + 1\right)}{\left(\sqrt{1 + 2^{-n} x_n} + 1\right)} \\ &= 2^{n+1} \frac{(1 + 2^{-n} x_n) - 1}{\sqrt{1 + 2^{-n} x_n} + 1} \\ &= \frac{2^{n+1} \cdot 2^{-n} x_n}{\sqrt{1 + 2^{-n} x_n} + 1} \\ &= \boxed{\frac{2x_n}{\sqrt{1 + 2^{-n} x_n} + 1}} \end{aligned}$$

The last expression avoids cancellation for large values of n.

(c) Calculate both roots of

$$3x^2 - 9^{14}x + 100 = 0$$

with 3 digit accuracy.

Solution:

By the quadratic formula, the roots are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 3, b = -9^{14}$, and $c = 100$. So that the two roots in exact arithmetic are

$$x_1 = \frac{9^{14} + \sqrt{9^{28} - 1200}}{6}$$

and

$$x_2 = \frac{9^{14} - \sqrt{9^{28} - 1200}}{6}$$

Using that $9^{28} \gg 1200$, the root x_1 can be approximated by

$$x_1 \approx \frac{2 \times 9^{14}}{6} = \boxed{7.63 \times 10^{12}}$$

Computing the second root using this form of the quadratic formula leads to loss of significance. By conjugate multiplication, we get

$$x_2 = \frac{200}{9^{14} + \sqrt{9^{28} - 1200}} \approx \frac{200}{2 \times 9^{14}} = \frac{100}{9^{14}} = \boxed{4.37 \times 10^{-12}}$$

(d) Find a good way of computing

$$f(x) = \frac{e^{2x} - 1}{2x}$$

for x near zero.

Solution:

Using Taylor's expansion around $x = 0$ gives

$$\begin{aligned} f(x) = \frac{e^{2x} - 1}{2x} &= \frac{\left(1 + (2x) + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \cdots\right) - 1}{2x} \\ &= \frac{2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \cdots}{2x} \\ &= \frac{2x + 2x^2 + \frac{4x^3}{3} + \cdots}{2x} \\ &= \boxed{1 + x + \frac{2}{3}x^2 + \cdots} \end{aligned}$$

2. (20 points)

(a) By forming a suitable function $f(x)$, show that the graphs of $u(x) = \frac{x}{2}$ and $v(x) = \tan^{-1}(x)$ intersect at three points: $x = \alpha$ in the interval $[2, 2.5]$, $x = 0$ and $x = -\alpha$.

Solution:

At the point of intersection, $u(x) = v(x)$, so that $f(x) := u(x) - v(x) = 0$. The points of intersection can be found by solving the equation $f(x) = 0$. First, let us determine the location of the roots. Note that $f(0) = \frac{0}{2} - \tan^{-1}(0) = 0$, so that $x = 0$ is a root. There is a root inside the interval $[2, 2.5]$ and another root in $[-2.5, -2]$. To see this, we apply the Intermediate Value Theorem

$$\begin{aligned} f(2.5) \cdot f(2) &= \left(\frac{2.5}{2} - \tan^{-1}(2.5)\right)\left(2 - \tan^{-1}(2)\right) = (0.0597)(-0.1071) < 0 \\ f(-2.5) \cdot f(-2) &= (-0.0597)(0.1071) < 0 \end{aligned}$$

Since the function $f(x)$ is odd, if α satisfies $f(\alpha) = 0$, then $f(-\alpha) = 0$.

(b) Perform 6 steps of the bisection method to estimate α.

Solution:

Starting with the interval $[a_0, b_0] = [2, 2.5]$, we have $c_0 = (2 + 2.5)/2 = 2.25$.

$$\begin{aligned} f(c_0) \cdot f(a_0) &= (-0.0276) \cdot (-1.071) > 0 : \text{ set } a_1 = c_0, b_1 = b_0 \\ c_1 &= (2.25 + 2.5)/2 = 2.375 \\ f(c_1) \cdot f(a_1) &= (0.0152)(-0.0276) < 0 : \text{ set } a_2 = a_1, b_2 = c_1 \\ c_2 &= (2.25 + 2.375)/2 = 2.3125 \\ f(c_2) \cdot f(a_2) &= (-0.064)(-0.0276) > 0 : \text{ set } a_3 = c_2, b_3 = b_2 \\ c_3 &= (2.3125 + 2.375)/2 = 2.3438 \\ f(c_3) \cdot f(a_3) &= (0.00438)(-0.0064) < 0 : \text{ set } a_4 = a_3, b_4 = c_3 \\ c_4 &= (2.3125 + 2.3438)/2 = 2.32815 \\ f(c_4) \cdot f(a_4) &= (-0.0010237)(-0.00639) > 0 : \text{ set } a_5 = c_4, b_5 = b_4 \\ c_5 &= (2.32815 + 2.3438)/2 = 2.335975 \\ f(c_5) \cdot f(a_5) &= (0.001673)(-0.001024) < 0 : \text{ set } a_6 = a_5, b_6 = c_5 \\ c_6 &= (2.32815 + 2.335975)/2 = 2.3320625. \end{aligned}$$

(c) Find an interval $I \subset \mathbb{R}$ so that the fixed-point iteration $x_{n+1} = 2\tan^{-1}(x_n)$ starting from any $x_0 \in I$ is guaranteed to converge to α

Solution:

Fixed point iteration converges on an interval $[a, b]$ if $|g'(x)| < 1$ for every $x \in [a, b]$. We solve for values of x such that $|g'(x)| < 1$. Since $g(x) = 2\tan^{-1}(x)$, it follows that $g'(x) = 2/(1 + x^2)$. Hence we need to solve $|2/(1 + x^2)| < 1$. The left hand side is always positive, so that $2/(1 + x^2) < 1$. Multiplying both sides by $1 + x^2$

yields $2 < 1 + x^2$. Hence $x^2 > 1$. The solutions are the intervals $x \in (-\infty, -1)$ and $x \in (1, \infty)$. Fixed point iteration starting with any $x_0 \in (1, \infty)$ converges to α (and to $-\alpha$ for any $x_0 \in (-\infty, -1)$).

(d) Perform 6 steps of the fixed-point iteration to approximate α starting with $x_0 = 2$.

Solution:

The fixed point iteration is of the form $x_{n+1} = g(x_n) = 2\tan^{-1}(x_n)$

$$\begin{aligned}
x_1 &= 2\tan^{-1}(2) = 2.2142974 \\
x_2 &= 2\tan^{-1}(2.2142974) = 2.293208 \\
x_3 &= 2\tan^{-1}(2.293208) = 2.319173 \\
x_4 &= 2\tan^{-1}(2.319173) = 2.327392 \\
x_5 &= 2\tan^{-1}(2.327392) = 2.329961 \\
x_6 &= 2\tan^{-1}(2.329961) = 2.330761.
\end{aligned}$$

(e) Explain why the fixed-point iteration $x_{n+1} = 2\tan^{-1}(x_n)$ is not guaranteed to converge to the root at $x = 0$.

Solution:

Fixed point iteration converges on an interval $[a, b]$ if $g([a, b]) \subset [a, b]$ and $|g'(x)| < 1$. Since $g'(0) = 2/(1 + 0^2) = 2 > 1$ in a neighborhood of $x = 0$, fixed point iteration does not converge to the root $x = 0$.

(f) For what starting values x_0 does Newton's method fail to find the roots of $f(x)$?

Solution:

Newton's iteration $x_{n+1} = x_n - f(x_n)/f'(x_n)$ breaks down when $f'(x_n) = 0$. Since $f(x) = x/2 - \tan^{-1}(x)$, we have $f'(x) = 1/2 - 1/(1 + x^2) = \frac{x^2-1}{2(1+x^2)}$, which has zeros at $x = \pm 1$. Hence Newton's iteration breaks down when $x_0 = \pm 1$.

3. (20 points)

(a) Verify that when Newton's method is used to compute $\sqrt[3]{a}$ (by solving the equation $x^3 = a$), the sequence of iterates is defined by

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right).$$

Solution:

Set $f(x) = x^3 - a$. Then $f'(x) = 3x^2$ and so Newton's iteration is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - a}{3x_n^2} \\ &= \frac{3x_n^3 - (x_n^3 - a)}{3x_n^2} \\ &= \frac{2x_n^3 + a}{3x_n^2} \\ &= \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right) \end{aligned}$$

(b) Perform three iterations of the scheme in part (a) for computing $\sqrt[3]{5}$, starting with $x_0 = 2$, and for the bisection method starting with the interval $[1, 2]$. How many iterations of the bisection method are required in order to obtain 10^{-6} accuracy?

Solution:

$$\begin{aligned} x_0 &= 2 \\ x_1 &= \frac{1}{3}\left(2 \cdot 2 + \frac{5}{2^2}\right) = \frac{21}{12} = 1.750000 \\ x_2 &= \frac{1}{3}\left(2 \cdot \frac{21}{12} + \frac{5}{(21/12)^2}\right) = \frac{502}{294} = 1.710884 \\ x_3 &= \frac{1}{3}\left(2 \cdot \frac{502}{294} + \frac{5}{(502/294)^2}\right) = \frac{908}{531} = 1.7099796. \end{aligned}$$

The correct value is $\sqrt[3]{5} \approx 1.7099759$ correct to 7 digits.

(c) Suppose $xe^x = 3$. By drawing suitable graphs, find a first approximation x_0 of the root of $xe^x = 3$ as the intersection of two graphs.

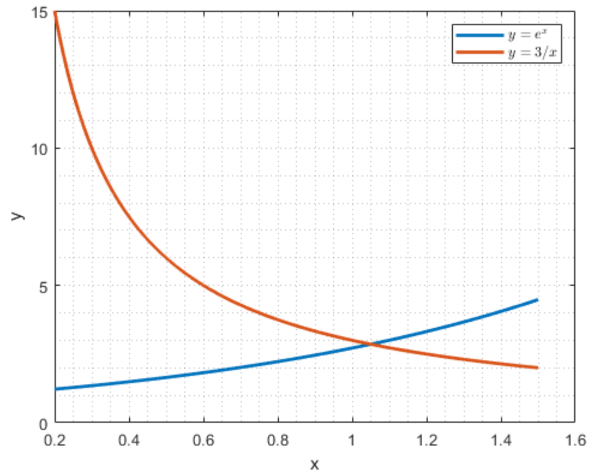
Solution:

The root of $xe^x = 3$ is the point of intersection of the graphs of $y = e^x$ and $y = 3/x$. The root is close to $x_0 = 1.0$.

(d) Hence use Newton's method to find the root of $xe^x = 3$ correct to 3 decimal places.

Solution:

Set $f(x) = xe^x - 3$. We want to use Newton's method to find the solution to



$f(x) = 0$. Note that $f'(x) = xe^x + e^x = e^x(x+1)$. Hence

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{x_n e_n^x - 3}{e_n^x (x_n + 1)} \\
x_0 &= 1 \\
x_1 &= 1 - \frac{1 \cdot e^1 - 3}{e^1 \cdot (1+1)} = 1.051819 \\
x_2 &= 1.049911 \\
x_3 &= 1.049909
\end{aligned}
$$

Hence the root is close to $x = 1.050$ to 3 decimal places.

(e) Explain what would happen if we were to choose $x_0 = -1$ as the first approximation in Newton's method for the root in (c).

Solution:

Note that $f'(x) = e^x(x+1) = 0$ at $x = -1$. Hence Newton's method breaks down immediately if $x_0 = -1$.