UNIVERSITY OF ZIMBABWE

BSc Honours in Mathematics & Computational Sciences: Level 2

NUMERICAL METHODS (MHTHCS212/HFM213/MHTH212)

July 2022
Time : 2 hours

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A6.

A1. Using Taylor's series, derive a formula so that the expression

$$\frac{\sin(2x)}{x}$$

can be evaluated accurately for x small on a digital computer. [5]

A2. The function $f(x) = 3x - e^{-2x}$ has a root close to $x = 0.2$

(a) Show that the equation $f(x) = 0$ has a root that lies between 0 and 0.5. [2]

(b) Use a fixed point iteration $x_{n+1} = g(x_n)$ for some g with $x_0 = 0.2$ to find x_1, x_2, and x_3 that approximate the root. Explain why fixed point iteration converges on $[0, 0.5]$. [5]

(c) Taking $x_0 = 0.2$ as the initial iterate, apply Newton's method **once** to find a second approximation x_1 giving your answer to three decimal places. [3]

(d) Explain what would happen if we had chosen $x_0 = \ln 3$ as the initial approximation in (c). [2]

A3. Using Taylor series, find the values of the coefficients α, β, and γ, the error term, and order p for the approximation formula

$$f'(x) \approx \frac{\alpha f(x+h) + \beta f(x) + \gamma f(x-h)}{h} + \mathcal{O}(h^p).$$

[5]

A4. Find the values of c_1, c_2 and c_3 such that the rule

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(-1) + c_2 f(0) + c_3 f(1)$$

has degree of precision greater than one. [5]

A5. Use a Newton's divided differences table method to find a polynomial that passes through the points (0,1), (2,3), and (3,0). [5]

A6. Consider the initial value problem: $y' = ty, \quad y(0) = 1$.

(a) Use separation of variables to show that the exact solution of the initial value problem is given by

$$y(t) = e^{t^2/2}.$$

[3]

(b) Apply Euler's method with step size $h = 0.1$ to the initial value problem to find an approximate solution and absolute error at $t = 0.2$ by comparing with the exact solution. [5]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B7 to B10.

B7. Consider the function $f(x) = 1/x$.

(a) Use Lagrange's interpolation to find a polynomial $p(x)$ of least degree that interpolates $f(x)$ at the points $x = 1, \frac{3}{2}, \frac{7}{4}$ and 2. [8]

(b) Use Newton's divided differences to show that the interpolating polynomial of least degree for the data is given by

$$\boxed{p(x) = 1 - \frac{2}{3}(x-1) + \frac{8}{21}(x-1)(x-1.5) - \frac{4}{21}(x-1)(x-1.5)(x-1.75)}$$

[10]

(c) Use the interpolating polynomial from part (b) to find an approximation $p(1.6)$ of $f(1.6)$. Compute the absolute error $|f(1.6) - p(1.6)|$. [4]

(d) The error form for the polynomial $p(x)$ of degree at most n interpolating a function $f \in C^{n+1}$ at points $x_0, \cdots, x_n \in [a, b]$ is given by

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\cdots(x - x_n)$$

for some $\xi \in (a, b)$. Determine the error form for the polynomial in part (b). [8]

B8. Consider the system of linear equations

$$\begin{cases} 2x - \ y \qquad = 1 \\ -x + 3y + \ z = 3 \\ \ x \qquad + 2z = 3 \end{cases}$$

(a) Rewrite the linear system in matrix-vector form $A\mathbf{x} = \mathbf{b}$. [2]

(b) Use Gaussian elimination to solve the linear system. [8]

(c) Use an LU factorization and back substitution to solve the above linear system. [9]

(d) Is the matrix A strictly diagonally dominant? If not, rewrite $A\mathbf{x} = \mathbf{b}$ in a strictly diagonally dominant form. [3]

(e) Apply **two** steps of the **Jacobi iteration** to the matrix system from part (d) to find an approximate solution of the linear system. [8]

B9. (a) Use the Trapezoidal rule with $n = 4$ sub-intervals to approximate the integral

$$\int_1^2 \ln(x)\, dx$$

[5]

(b) The error term for the composite Trapezoidal rule to approximate $\int_a^b f(x)\, dx$ is given by

$$-\frac{(b-a)h^2}{12} f^{(2)}(c)$$

for some $c \in (a, b)$ with h the width of each sub-interval. Use the error term to determine the maximum value of h such that Simpson's rule approximates the integral

$$\int_1^2 \ln(x)\, dx$$

to an error of less than 10^{-3}. [5]

(c) For this value of h from part (b), what is the corresponding number n of sub-intervals? [2]

(d) Derive the **composite left endpoint** rule, and show that the error term is first order. That is, show that

$$\int_a^b f(x)\, dx = h \sum_{\ell=0}^{n-1} f(a + \ell h) + \frac{(b-a)h}{2} f'(c),$$

where $c \in (a, b)$. [10]

(f) Using Gaussian quadrature with three nodes, approximate the integral

$$\int_1^2 \ln(x)\, dx.$$

[8]

Hint: nodes $\left\{-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}\right\}$ and weights $\left\{\frac{5}{9}, \frac{8}{9}, \frac{5}{9}\right\}$

B10. (a) For the initial value problem $y' = f(t, y)$, with $y(t_0) = y_0$, derive Heun's scheme

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_n + h, y_n + hk_1)\right], \qquad k_1 = f(t_n, y_n)$$

by integrating the differential equation on $[t_n, t_{n+1}]$ and using the trapezoidal rule to approximate the integral. [8]

(b) Consider the initial value problem:

$$y' = ty, \quad t \in [0, 1], \quad y(0) = 1.$$

The analytical solution of the above differential equation is

$$y(t) = e^{t^2/2}.$$

(i) Use **one step** of Heun's scheme with $h = 0.1$ to approximate the solution at $t = 0.1$. [8]

(ii) Use **one step** of the Runge-Kutta (order 4) method with $h = 0.1$ to approximate the solution at $t = 0.1$. [10]
Hint: $k_1 = f(t_n, y_n)$, $k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$, $k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$, $k_4 = f(t_n + h, y_n + hk_3)$

$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

(iii) Compute the absolute errors of the two methods by comparing your results with the analytical solution. Which is more accurate? [4]

END OF QUESTION PAPER