Fourier Series Tutorial 1

MTE201 Engineering Mathematics 2

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1. For $n, m \geq 1$ show that
$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \pi, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

   Solution: We use the trig. identity
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
   Setting $\alpha = nx$, and $\beta = mx$, we get
$$\sin(nx)\sin(mx) = \frac{1}{2}\left[\cos(n-m)x - \cos(n+m)x\right].$$
   So that
$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos(n-m)x\,dx - \int_{-\pi}^{\pi} \cos(n+m)x\,dx\right]$$
   Suppose $n = m \geq 1$. Then
$$\frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(0)\,dx = \pi$$
   and
$$\int_{-\pi}^{\pi} \cos(2nx)\,dx = \frac{1}{2n}\sin(nx)\Big|_{-\pi}^{\pi} = 0$$
   Thus, by the above calculation, if $n = m$, we have
$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx &= \int_{-\pi}^{\pi} \sin^2(nx)\,dx \\ &= \pi.\end{aligned}$$
   If $n \neq m$, we have
$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx &= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos(n-m)x\,dx - \int_{-\pi}^{\pi} \cos(n+m)x\,dx\right] \\ &= \frac{1}{2}\frac{\sin(n-m)x}{n-m}\Big|_{-\pi}^{-\pi} - \frac{1}{2}\frac{\sin(n+m)x}{n+m}\Big|_{-\pi}^{-\pi} \\ &= 0.\end{aligned}$$

because $(n-m)$ and $(n+m)$ are integers, and we know that $\sin(k\pi)=0$ for any integer k. Finally, we can write our solution as

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx \quad = \quad \begin{cases} \pi, & \text{if } n=m \\ 0, & \text{if } n\neq m \end{cases}$$

2. For $n, m \geq 1$ show that

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0.$$

3. Find the Fourier series of the following functions. Determine first if the function is even, odd, or neither, and proceed to find an appropriate series.

   (a)
   $$f(x) = |\sin(x)|, \quad \text{on } [-\pi,\pi]$$

   (b)
   $$f(x) = \begin{cases} e^x, & \text{if } 0 \leq x \leq 2, \\ 0, & \text{if } -2 \leq x < 0. \end{cases}$$

   (c)
   $$f(x) = x+\pi, \quad \text{on } [-\pi,\pi].$$

   Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,dx = \frac{1}{\pi}\frac{x^2}{2}\bigg|_{-\pi}^{\pi} + x\bigg|_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}-\frac{\pi^2}{2}\right)+2\pi \\ &= 2\pi \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx \end{aligned}$$

   Observe that $x\cos nx$ is an odd function on the interval $[-\pi,\pi]$. Hence the first integral disappears, and we have left

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\pi\cos nx\,dx = \int_{-\pi}^{\pi}\cos nx\,dx = \frac{\sin nx}{n}\bigg|_{-\pi}^{\pi} = 0.$$

   Using that $x\sin(nx)$ is even on $[-\pi,\pi]$ and that $\pi\sin(nx)$ is odd, we get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x\sin(nx)\,dx$$

Integrating by parts, we take $u = x$, and $dv = \sin(nx)dx$. Then $du = dx$ and $v = -\frac{\cos(nx)}{n}$.

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n}\bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}\,dx\right] \\
&= \\
&= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left(-\frac{2\pi}{n}(-1)^n + 0\right) \\
&= \frac{2}{n}(-1)^{n+1}
\end{aligned}
$$

Therefore the Fourier series is (remember that $a_n = 0$ for $n = 1, 2, 3, \cdots$

$$
\begin{aligned}
f(x) &= \frac{2\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx) \\
&= \pi + \frac{2}{1}\sin(x) - \frac{2}{2}\sin(2x) + \frac{2}{3}\sin(3x) - \frac{2}{4}\sin(4x) + \frac{2}{5}\sin(5x) + \cdots
\end{aligned}
$$

(d)

$$
f(x) = \begin{cases} x^2, & \text{if } 0 < x < \pi \\ 0, & \text{if } -\pi < x < 0. \end{cases}
$$

4. Use the result of Problem 3(c) to show that

$$
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
$$

Solution:

Let us take $x = \frac{\pi}{2}$. Then

$$
f(x) = f(\pi/2) = \frac{\pi}{2} + \pi = \frac{3\pi}{2}
$$

Substituting $\pi/2$ into the Fourier series obtained in Part 3(c), we get

$$
\begin{aligned}
f\left(\frac{\pi}{2}\right) &= \pi + 2\sin(\pi/2) - \sin(2\pi/2) + \frac{2}{3}\sin(3\pi/2) - \frac{1}{2}\sin(4\pi/2) + \frac{2}{5}\sin(5\pi/2) - \frac{2}{6}\sin(6\pi/2) + \cdots \\
&= \pi + 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots
\end{aligned}
$$

We have used that $\sin(m\pi/2) = 0$ if m is even. Since we know that $f(\pi/2) = \frac{3\pi}{2}$, we have

$$\frac{3\pi}{2} = \pi + 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

From which we obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

as required.

5. Use the result of Problem 3(d) to show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$