Fourier Series Fact Sheet

MT201: Engineering Mathematics 2

September 2022

1 Trigonometric Identities

$$\cos(n\pi) = (-1)^n, \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos^2(\alpha) = \frac{1}{2}\left(1 + \cos(2\alpha)\right)$$

$$\sin^2(\alpha) = \frac{1}{2}\left(1 - \cos(2\alpha)\right)$$

2 Complex Variables

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

3 Fourier Series:

3.1 General Fourier Series

Fourier series on an interval $[-L, L]$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

where the Fourier coefficients are computed as:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\, dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\, dx.$$

3.2 Fourier Series of an Even Function

If the function f is even, i.e. $f(x) = f(-x)$, then the graph of $f(x)$ is symmetric about the y-axis (e.g. the graphs of x^2, $\cos(x)$, $|\sin(x)|$). Then the Fourier series of an even function can be simplified as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

where

$$a_0 = \frac{2}{L}\int_{0}^{L} f(x)\, dx,$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi}{L}x\, dx$$

3.3 Fourier Series of an Odd Function

If the function is odd, i.e. $f(-x) = -f(x)$, then the graph of f is flipped around the x-axis across the point $x = 0$. (e.g. the graphs of x, x^3, $\sin(x)$). Then the Fourier series of an odd function can be simplified as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

where,

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi}{L}x\, dx$$

4 Fourier Transforms

Let $f(x)$ be a function such that $f, f' \to 0$ as $|x| \to \infty$. Then the Fourier Transform of $f(x)$ is given by

$$\widehat{f}(\mu) := \mathcal{F}(\mu) := \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\mu x}\, dx$$

The Inverse Fourier Transform is

$$\mathcal{F}^{-1}[\widehat{f}(\mu)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(\mu) e^{i\mu x} \, d\mu$$

Let $f^{(n)}(x)$ be the n-th derivative of $f(x)$ with respect to x. Then

$$\mathcal{F}[f^{(n)}(x)] = (i\mu)^n \mathcal{F}[f(x)]$$

The Fourier Sine Transform is

$$\widehat{f}_S(\mu) := \mathcal{F}_S(\mu) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\mu x) \, dx$$

The Inverse Fourier Sine Transform is

$$\mathcal{F}_S^{-1}[\widehat{f}_S(\mu)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \widehat{f}_S(\mu) \sin(\mu x) \, d\mu$$

The Fourier Cosine Transform is

$$\widehat{f}_C(\mu) := \mathcal{F}_C(\mu) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\mu x) \, dx$$

The Inverse Fourier Cosine Transform is

$$\mathcal{F}_C^{-1}[\widehat{f}_C(\mu)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \widehat{f}_C(\mu) \cos(\mu x) \, d\mu$$

The Fourier Sine and Cosine Transforms of $f''(x)$ are:

$$\mathcal{F}_S[f''(x)] = \sqrt{\frac{2}{\pi}} \mu f(0) - \mu^2 \mathcal{F}_S[f(x)]$$

$$\mathcal{F}_C[f''(x)] = -\sqrt{\frac{2}{\pi}} f'(0) - \mu^2 \mathcal{F}_C[f(x)]$$