Fourier Transforms - PDE

MT201: Engineering Mathematics 2

September 2022

1. Use Fourier Transforms to find the solution $u(x,t)$ of the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad \alpha > 0$$
$$u(x,0) = f(x).$$

where

$$u(x,0) = f(x) = \begin{cases} 1, & \text{if } |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

Since the domain is all the real numbers, we can use the full Fourier transform. Taking the Fourier Transform (with respect to the x-variable) on both sides of the PDE

$$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \alpha \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

The Fourier series of the right hand side is just

$$\alpha \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) = \alpha (i\mu)^2 \mathcal{F}(u) = -\alpha \mu^2 \mathcal{F}(u)$$

To simplify notation, denote by $\widehat{u}(\mu,t) = \mathcal{F}[u(x,t)]$. We have

$$\frac{\partial \widehat{u}(\mu,t)}{\partial t} = -\alpha \mu^2 \widehat{u}(\mu,t)$$

This is an ODE in the variable t, which has the general solution

$$\widehat{u}(\mu,t) = C(\mu) e^{-\alpha \mu^2 t}$$

where $C(\mu)$ depends on the initial conditions. When $t = 0$, we get

$$\widehat{u}(x,0) = C(\mu).$$

To figure out the value of $C(\mu)$, take the Fourier transform of the initial condition

$$u(x,0) = f(x) = \begin{cases} 1, & \text{if } |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
\mathcal{F}[u(x,0)] = \mathcal{F}[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\mu x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} 1 \cdot e^{i\mu x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-2}^{0} e^{i\mu x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{i\mu x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{-i\mu x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{i\mu x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{2} (e^{i\mu x} + e^{-i\mu x})\, dx \\
&= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{2} \frac{e^{i\mu x} + e^{-i\mu x}}{2}\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{2} \cos(\mu x)\, dx \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin(\mu x)}{\mu} \Big|_{0}^{2} \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin(2\mu)}{\mu}
\end{aligned}$$

We have thus shown that

$$\widehat{u}(\mu, 0) = \sqrt{\frac{2}{\pi}} \frac{\sin(2\mu)}{\mu}.$$

So that

$$\widehat{u}(\mu, 0) = C(\mu) = \sqrt{\frac{2}{\pi}} \frac{\sin(2\mu)}{\mu}.$$

Plugging in the value of $C(\mu)$ yields

$$\widehat{u}(\mu, t) = \sqrt{\frac{2}{\pi}} \frac{\sin(2\mu)}{\mu} e^{-\alpha\mu^2 t}.$$

This is the Fourier Transform of the solution $u(x,t)$. To get the solution

$u(x,t)$, we need to find the Inverse Fourier Transform of $\widehat{u}(\mu, t)$.

$$\begin{aligned}
u(x,t) = \mathcal{F}^{-1}(\widehat{u}(\mu,t)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{u}(\mu,t)e^{-i\mu x}\, d\mu \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}e^{-i\mu x}\, d\mu \\
&= \frac{1}{\pi}\int_{-\infty}^{0} \frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}e^{-i\mu x}\, d\mu + \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}e^{-i\mu x}\, d\mu \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}e^{i\mu x}\, d\mu + \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}e^{-i\mu x}\, d\mu \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}(e^{i\mu x} + e^{-i\mu x})\, d\mu \\
&= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}\frac{e^{i\mu x} + e^{-i\mu x}}{2}\, d\mu \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}\cos(\mu x)\, d\mu
\end{aligned}$$

Therefore the solution is

$$\boxed{u(x,t) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin(2\mu)}{\mu}e^{-\alpha\mu^2 t}\cos(\mu x)\, d\mu}$$

2. Use a Fourier Transform to find the solution of the half range heat equation.

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \;\; t > 0 \\
&u(x,0) = f(x), \\
&u(0,t) = 0.
\end{aligned}$$

where

$$f(x) = e^{-x}$$

Solution:

Since the interval is $0 < x < \infty$, we use either a Fourier Sine Transform or a Fourier Cosine Transform. In this case, the initial value $u(0,t) = f(x)$ is given, hence we use a Fourier Sine Transform.
If the initial flux $\frac{\partial u}{\partial x}(0,t)$ was provided, we would have used the Fourier Cosine Transform.
Denote by $\widehat{u}_S(\mu,t) = \mathcal{F}_S[f(x)]$ the Fourier Sine Transform. Taking the Fourier Sine Transfoorm on both sides of the equation, we get

$$\frac{\partial}{\partial t}\widehat{u}_S(\mu,t) = \alpha\mathcal{F}_S\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)$$

The right hand side evaluates to

$$\alpha \mathcal{F}_S \left(\frac{\partial^2}{\partial x^2} u(x,t) \right) = \sqrt{\frac{2}{\pi}} \alpha\mu \; u(0,t) - \alpha\mu^2 \widehat{u}_S(\mu,t)$$

Since $u(0,t) = 0$, it follows that

$$\alpha \mathcal{F}_S \left(\frac{\partial^2}{\partial x^2} u(x,t) \right) = -\alpha\mu^2 \widehat{u}_S(\mu,t).$$

It follows that

$$\frac{\partial}{\partial t} \widehat{u}_S(\mu,t) = -\alpha\mu^2 \widehat{u}_S(\mu,t).$$

This is an ODE with solution

$$\widehat{u}_S(\mu,t) = C(\mu) e^{-\alpha\mu^2 t}$$

At $t = 0$,

$$\widehat{u}(\mu,0) = C(\mu).$$

The value of $C(\mu)$ is determined by taking the Fourier Sine Transform of the initial condition,

$$\mathcal{F}_S[f(x)] = \mathcal{F}_S[e^{-x}].$$

To determine this, note that

$$f''(x) = f(x)$$

for

$$f(x) = e^{-x}.$$

Using that $f(0) = e^{-0} = 1$ we get

$$\sqrt{\frac{2}{\pi}} \mu f(0) - \mu^2 \mathcal{F}[f(x)] = \mathcal{F}[f(x)]$$

$$\mathcal{F}[f(x)] = \sqrt{\frac{2}{\pi}} \frac{\mu}{1+\mu^2}.$$

This shows that

$$C(\mu) = \sqrt{\frac{2}{\pi}} \frac{\mu}{1+\mu^2}.$$

The Fourier Sine Transform of the solution is

$$\widehat{u}_S(\mu,t) = \sqrt{\frac{2}{\pi}} \frac{\mu}{1+\mu^2} e^{-\alpha\mu^2 t}.$$

To determine the solution, take the inverse Fourier Sine Transform

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\mu}{1+\mu^2} e^{-\alpha \mu^2 t} \sin(\mu x)\, d\mu \\ &= \frac{2}{\pi} \int_0^\infty \frac{\mu}{1+\mu^2} e^{-\alpha \mu^2 t} \sin(\mu x)\, d\mu \end{aligned}$$

$$\boxed{u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{\mu}{1+\mu^2} e^{-\alpha \mu^2 t} \sin(\mu x)\, d\mu}$$