University Of Zimbabwe

MTE201: Engineering Mathematics 2
Section: Fourier Series and Partial Differential Equations
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Lecturer:
Dr. S. Kapita

Department:
Mathematics and Comuputational Sciences

Course Outline

- Fourier series
- Orthogonal relations and the Euler formula for the coefficients
- Bessel's inequality and Parseval's identity
- Fourier series of even and odd functions
- The Fourier transform, inverse transform, convolutions
- Application of Fourier series and transforms to Partial Differential Equations.

Contents

1 Introduction to Fourier Series

A function f of a real variable is said to be **periodic** with period $2L$ if

$$f(x+2L) = f(x)$$

holds for all $x \in \mathbb{R}$.
A function f is said to satisfy the **Dirichlet conditions** if the following three conditions hold for f on any one period (e.g. on $[-L, L]$ or $[0, 2L]$ etc)

1. The function $|f(x)|$ can be integrated over a period and the integral is finite.
2. The function f has a finite number of maxima and minima in any one period.
3. The function f has a finite number of discontinuities in any one period, and the discontinuities are finite.

If f is a periodic function that satisfies the Dirichlet conditions, then at any point x where f is continuous, $f(x)$ is equal to the sum of its Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

where the coefficients (amplitudes) a_n and b_n for frequencies $n = 0, 1, 2, \cdots$ depend on the function f, and will be derived in the next section.

If $f(x)$ has a discontinuity at a point x, then the Fourier series will converge to the average of the left limit $f(x^-)$ and the right limit $f(x^+)$

$$\frac{f(x^-) + f(x^+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

When the period of the function is 2π then the Fourier series can be simplified:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

1.1 Orthogonality

There is a nice integral formula for finding the coefficients a_n and b_n of a Fourier series. This is based on the **orthogonality** of the functions

$$\cos \frac{n\pi x}{L}, \quad \sin \frac{m\pi x}{L}$$

for different values of $n, m = 0, 1, 2, \cdots$

Theorem 1.1 (Orthogonality Relations) *If* $m, n = 0, 1, 2, 3, \cdots$, *then*

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} L\delta_{n,m} & \text{if } m,n \geq 1 \\ 2L, & \text{if } n = m = 0. \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} L\delta_{n,m} & \text{if } m,n \geq 1 \\ 0, & \text{if } n = m = 0. \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0 \quad \text{for all } m,n = 0,1,2,\cdots.$$

where

$$\delta_{n,m} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Assignment.

1.2 Euler Formula for the Coefficients

Recall the Fourier series equation:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

In this section, we will assume that the interval is fixed at $[-L, L]$. To find the values of the coefficients a_0, a_n and b_n, simply multiply both sides of the equation by $\cos\frac{n\pi x}{L}$ or $\sin\frac{n\pi x}{L}$ for any $n = 0, 1, 2, 3, \cdots$ and integrate from $-L$ to L. Then using the orthogonality relations, we get

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \\ a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \end{aligned}$$

Remark: Some authors write the Fourier series as

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

and the coefficient a_0 as

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$$

This is just a matter of convention, the result is the same. Make sure to stick to one convention!

Example 1: Find the Fourier series representation of the periodic function

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x < 0 \\ 1, & \text{if } 0 \leq x < \pi. \end{cases}$$

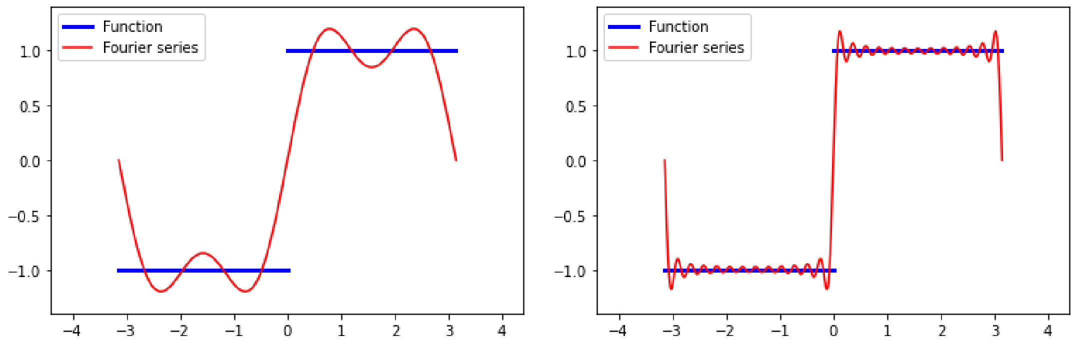
Solution:

We need to determine the coefficients a_0, a_n and b_n

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi} 1\ dx + \int_{-\pi}^{0} -1\ dx\right] \\ &= \frac{1}{\pi}[\pi - \pi] \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi} \cos nx dx - \int_{-\pi}^{0} \cos nx dx\right] \\ &= \frac{1}{n\pi}\sin nx\Big|_0^{\pi} - \frac{1}{n\pi}\sin nx\Big|_{-\pi}^{0} \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi} \sin nx dx - \int_{-\pi}^{0} \sin nx dx\right] \\ &= -\frac{1}{n\pi}\cos nx\Big|_0^{\pi} + \frac{1}{n\pi}\cos nx\Big|_{-\pi}^{0} \\ &= -\frac{1}{n\pi}(\cos n\pi - \cos 0) + \frac{1}{n\pi}(\cos 0 - \cos(-n\pi)) \\ &= -\frac{1}{n\pi}((-1)^n - 1) + \frac{1}{n\pi}(1 - (-1)^n) \\ &= \frac{2}{n\pi}(1 - (-1)^n) \\ &= \begin{cases} \frac{4}{n\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$



(a) (b)

Figure 1: (a) function in Example 1 and Fourier series with $n = 3$ terms (b) $n = 25$ terms. Note the overshooting of the partial Fourier series near the points of discontinuity.

Therefore

$$\begin{aligned} f(x) &= \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}. \end{aligned}$$

Example 2:

Find the Fourier series representation of the periodic function

$$f(x) = x^2, \quad -\pi < x < \pi.$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{3\pi}x^3\Big|_{-\pi}^{\pi} = \frac{2}{3}\pi^2.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx.$$ We now apply integration by parts to the integral.

Let

$$u = x^2,\ du = 2xdx,\ dv = \cos nx dx,\ v = \frac{1}{n}\sin nx.$$

Thus

$$\begin{aligned} a_n &= \frac{1}{\pi}\frac{x^2}{n}\sin nx\Big|_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx dx \\ &= -\frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx dx. \end{aligned}$$ Applying integration by parts again,

we have

$$u = x,\ du = dx,\ dv = \sin nx dx,\ v = -\frac{1}{n}\cos nx.$$

Thus

$$\begin{aligned} a_n &= -\frac{2}{n\pi}\left[-\frac{x}{n}\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx dx\right] \\ &= \frac{4}{n^2}(-1)^n \\ b_n &= 0 \ \ \big(\text{since } x^2 \sin nx \ \text{ is odd}\big) \end{aligned}$$

The Fourier series for f is then

$$\begin{aligned} f(x) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cos nx \\ &= \frac{\pi^2}{3} - 4\cos x + 4\frac{\cos 2x}{2^2} - 4\frac{\cos 3x}{3^2} + \cdots . \end{aligned}$$

Fourier series can be used to find the values of certain infinite series. For example, what is the value of the infinite sum?

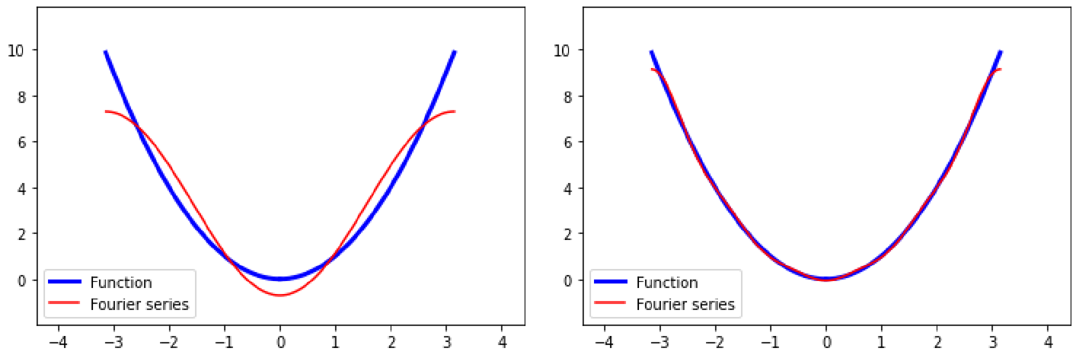
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

To find this sum, suppose that $f(x) = x^2$ and $x = \pi$. Then $f(x) = f(\pi) = \pi^2$. Substituting this into the Fourier series above for $f(x) = x^2$, we get

$$\begin{aligned} f(\pi) = \pi^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cos n\pi \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^{2n} \frac{4}{n^2} \quad (\text{ since } \ \cos n\pi = (-1)^n) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2} \end{aligned}$$

Rearranging the last expression gives the formula:

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$



(a) (b)

Figure 2: (a) $f(x) = x^2$ and Fourier series with $n = 1$ (b) $n = 5$

1.3 Bessel's Inequality and Parseval's Identity

Let us consider the n^{th} partial sum

$$S_n(x) = \frac{a_0}{2} + \sum_{\ell=1}^{n} \left(a_\ell \cos \frac{\ell \pi x}{L} + b_\ell \sin \frac{\ell \pi \ell}{L} \right),$$

of the Fourier series of $f(x)$ defined over the interval $-L \leq x \leq L$. Then provided the integral $\int_{-L}^{L} [f(x)]^2 \, dx$ exists and is finite, we have the result

$$\int_{-L}^{L} [f(x) - S_n(x)]^2 \, dx = \int_{-L}^{L} [f(x)]^2 \, dx - 2 \int_{-L}^{L} f(x) S_n(x) dx + \int_{-L}^{L} [S_n(x)]^2 \, dx$$

From the definition of the Fourier partial sum $S_n(x)$ it follows that

$$\int_{-L}^{L} [S_n(x)]^2 \, dx = \int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{\ell=1}^{n} \left(a_\ell \cos \frac{\ell \pi x}{L} + b_\ell \sin \frac{\ell \pi x}{L} \right) \right]^2 dx.$$

The orthogonality of the sine and cosine functions reduces this to

$$\begin{aligned} \int_{-L}^{L} [S_n(x)]^2 \, dx &= \int_{-L}^{L} \frac{a_0^2}{4} dx + \sum_{\ell=1}^{n} \left[a_\ell^2 \int_{-L}^{L} \cos^2 \frac{\ell \pi x}{L} dx \right] + \sum_{\ell=1}^{n} \left[b_\ell^2 \int_{-L}^{L} \sin^2 \frac{\ell \pi x}{L} dx \right] \\ &= L \left[\frac{a_0^2}{2} + \sum_{\ell=1}^{n} \left(a_\ell^2 + b_\ell^2 \right) \right] \end{aligned}$$

If $f(x)$ is replaced by its Fourier series, a similar argument shows that

$$\int_{-L}^{L} f(x) S_n(x) dx = L \left[\frac{a_0^2}{2} + \sum_{\ell=1}^{n} \left(a_\ell^2 + b_\ell^2 \right) \right]$$

Combining the terms above gives

$$\int_{-L}^{L} [f(x) - S_n(x)]^2 \, dx = \int_{-L}^{L} [f(x)]^2 \, dx - L \left[\frac{a_0^2}{2} + \sum_{\ell=1}^{n} \left(a_\ell^2 + b_\ell^2 \right) \right]$$

The integral on the left is non-negative, because its integrand is a squared quantity, so it follows at once that for all n

$$\frac{a_0^2}{2} + \sum_{\ell=1}^{n} \left(a_\ell^2 + b_\ell^2 \right) \leq \frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx.$$

This is **Bessel's Inequality** for Fourier series. When $f(x)$ is square-integrable, i.e. $\int_{-L}^{L} [f(x)]^2 \, dx$ exists and is finite, then the series

$$\frac{a_0^2}{2} + \sum_{\ell=1}^{n} \left(a_\ell^2 + b_\ell^2 \right)$$

is convergent. Thus, the coefficients in the associated Fourier series must be such that

$$\lim_{n \to \infty} a_n = 0 \quad \text{and} \quad \lim_{n \to \infty} b_n = 0.$$

If the n^{th} partial sum $S_n(x)$ converges to $f(x)$ in the sense that

$$\lim_{n \to \infty} \int_{-L}^{L} [f(x) - S_n(x)]^2 \, dx = 0,$$

then

$$\frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{\ell=1}^{n} \left(a_\ell^2 + b_\ell^2 \right).$$

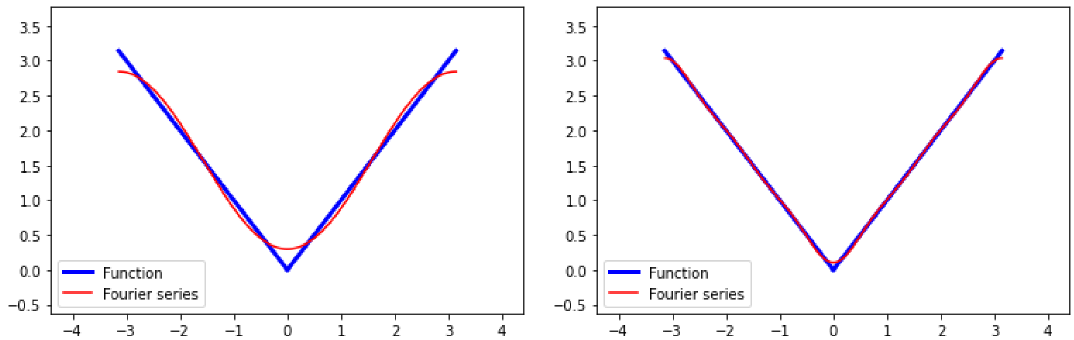
This is known as the **Parseval identity** for the Fourier series.

1.4 Fourier Series of Even and Odd Functions

Recall that a function $f(x)$ is **even** if $f(-x) = f(x)$ and it is **odd** if $f(-x) = -f(x)$ for every x in the domain.

If $f(x)$ is an even function defined on the interval $-L \leq x \leq L$, then

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad \text{with} \\ a_0 &= \frac{2}{L} \int_0^L f(x) dx \\ a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad \text{for } n = 1, 2, \cdots. \end{aligned}$$



(a) (b)

Figure 3: (a) $f(x) = |x|$ and Fourier series with $n = 1$ (b) $n = 5$

If $f(x)$ is an odd function, then

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad \text{with} \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad \text{for } n = 1, 2, \cdots . \end{aligned}$$

Example 3:

Find the Fourier series representation of

$$f(x) = |x|, \quad \text{on the interval } -L \le x \le L$$

Solution

The function $f(x) = |x|$ is defined as:

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \ge 0 \end{cases}$$

The Fourier coefficients of $f(x)$ are calculated as follows:

$$a_0 = \frac{2}{L}\int_0^L x\,dx = L$$

and

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx \ \text{(integration by parts)} \\
&= \frac{2}{L}\left[\frac{L^2\cos\frac{n\pi x}{L}}{n^2\pi^2}\right]\Bigg|_0^L \\
&= \frac{2L}{n^2\pi^2}\Big[(-1)^n - 1\Big], \quad \text{for } n = 1, 2, \cdots \\
&= \begin{cases} -\frac{4L}{n^2\pi^2}, & \text{when } n \text{ is odd} \\ 0, \text{when } n \neq 0, \text{ is even.} & \end{cases}
\end{aligned}$$

Thus, the Fourier series representation of $f(x) = |x|$ for $-L \leq x \leq L$ is

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\left[\frac{\cos\frac{\pi x}{L}}{1^2} + \frac{\cos\frac{3\pi x}{L}}{3^2} + \frac{\cos\frac{5\pi x}{L}}{5^2} + \cdots\right].$$

The sequence of positive odd numbers can be written in the form $2n-1$ with $n = 1, 2, \cdots$, so the last result can be expressed more concisely as

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}, \quad \text{for} - L \leq x \leq L.$$

Example 4:
Find the Fourier series representation of $f(x) = x$ on the interval $-2 \leq x \leq 2$

Solution
Using the fact that $L = 2$,

$$b_n = \int_0^2 x\sin\frac{n\pi x}{2}dx = -\frac{4}{n\pi}\cos n\pi = \frac{4}{n\pi}(-1)^{n+1}$$

and as the function is odd all the coefficients $a_n = 0$. The required Fourier series representation is

$$f(x) = \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right],$$

which can be written in the more concise form

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}, \quad \text{for} - 2 \leq x \leq 2.$$

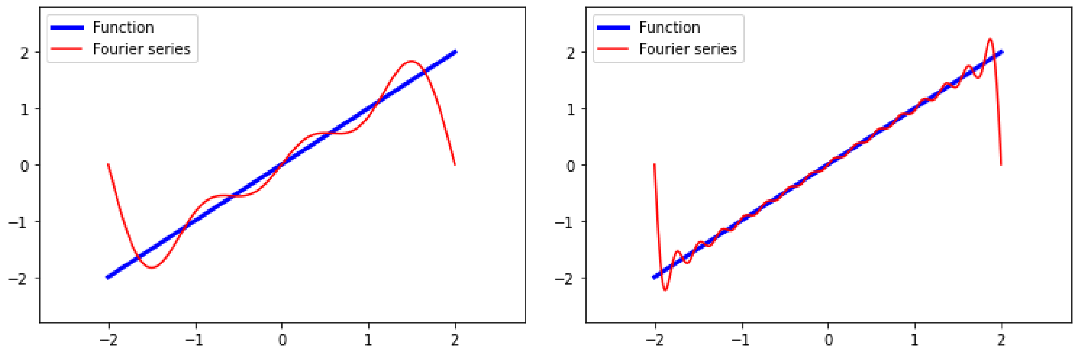


Figure 4: (a) Example 4: $f(x) = x$ and Fourier series on the interval $[-2, 2]$ with $n = 3$ (b) $n = 15$

2 The Fourier Transform

2.1 Fourier transforms as integrals

There are several ways to define the Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$. In this section, we define it using an integral representation and state some basic uniqueness and inversion properties, without proof.

Definition 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$. The Fourier transform of $f(x)$, denoted by $F(\mu) = \mathcal{F}[f(x)]$, is given by the integral*

$$F(\mu) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\mu x}dx,$$

for $x \in \mathbb{R}$ for which the integral exists.

We have the **Dirichlet condition** *for inversion of Fourier integrals.*

Theorem:
Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that

1. *$\int_{-\infty}^{\infty} |f(x)|dx$ converges and*
2. *in any finite interval, f, f' are piece-wise continuous with at most finitely many maxima/minima/discontinuities.*

Let $F(\mu) = \mathcal{F}[f(x)]$. Then if $f(x)$ is continuous at $x \in \mathbb{R}$, we have the **Fourier Inversion** *formula*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\mu)e^{ix\mu}d\mu.$$

Moreover, if f is discontinuous at $x \in \mathbb{R}$ and $f(x^+)$ and $f(x^-)$ denote the right and left limits of f at x, then

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\mu) e^{ix\mu} d\mu.$$

Note that the formula above automatically holds if f is continuous at x, since $f(x^+) = f(x^-)$.

Example 1:

Find the Fourier transform of $f(x) = e^{-|x|}$ and hence using inversion, deduce that

$$\int_0^{\infty} \frac{d\mu}{1+\mu^2} = \frac{\pi}{2} \quad \text{and} \quad \int_0^{\infty} \frac{\mu \sin \mu x}{1+\mu^2}\, d\mu = \frac{\pi}{2} e^{-x}, \; x > 0.$$

Solution:
We write

$$\begin{aligned}
F(\mu) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\mu x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x} e^{-i\mu x}\, dx + \int_0^{\infty} e^{-x} e^{-i\mu x}\, dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(1-i\mu)x} dx + \int_0^{\infty} e^{-(1+i\mu)x} dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\left. \frac{e^{(1-i\mu)x}}{1-i\mu} \right|_{-\infty}^{0} - \left. \frac{e^{-(1+i\mu)x}}{1+i\mu} \right|_{0}^{\infty} \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-i\mu} + \frac{1}{1+i\mu} \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{(1+i\mu) + (1-i\mu)}{(1-i\mu)(1+i\mu)} \right] \\
&= \frac{2}{\sqrt{2\pi}} \frac{1}{1+\mu^2} \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{1}{1+\mu^2} \right].
\end{aligned}$$

Now by the inversion formula,

$$
\begin{aligned}
e^{-|x|} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\mu) e^{ix\mu} d\mu \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+\mu^2} e^{ix\mu} d\mu \\
&= \frac{1}{\pi} \left[\int_{-\infty}^{0} e^{ix\mu} \frac{1}{1+\mu^2}\, d\mu + \int_{0}^{\infty} e^{ix\mu} \frac{1}{1+\mu^2}\, d\mu \right] \\
&= \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-ix\mu} + e^{ix\mu}}{1+\mu^2}\, d\mu \quad \left(\text{since } \cos u = \frac{1}{2}\left(e^{iu} + e^{-iu}\right) \right) \\
&= \frac{2}{\pi} \int_{0}^{\infty} \frac{\cos \mu x}{1+\mu^2} d\mu.
\end{aligned}
$$

now this formula holds at $x = 0$, so substituting $x = 0$ into the above gives the first required identity. Differentiating under the integral with respect to x as we may for $x > 0$, gives the second required identity.

2.2 Properties of the Fourier transform

Recall the definition of Fourier transform:

$$
\mathcal{F}[f(x)] = F(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx
$$

1. $\mathcal{F}[\cdot]$ is a linear operator. For $a, b \in \mathbb{R}$ we have

$$
\mathcal{F}[af_1(x) + bf_2(x)] = a\mathcal{F}[f_1(x)] + b\mathcal{F}[f_2(x)] = aF_1(\mu) + bF_2(\mu)
$$

2. the Fourier Transform exchanges differentiation with multiplication: $\mathcal{F}[f'(x)] = i\mu F(\mu)$.
   In general $\mathcal{F}[f^n(x)] = (i\mu)^n F(\mu)$.

Example 2:

Find the Fourier transform of the function

$$
f(x) = \begin{cases} e^{-x} \sin x, & \text{if } x > 0 \\ \\ 0, & \text{if } x < 0. \end{cases}
$$

Solution:

(complete the missing steps)

$$\begin{aligned}
F(\mu) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x} \sin x e^{-i\mu x} dx \\
&= \\
&. \\
&. \\
&. \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1 + (1 + i\mu)^2} \right].
\end{aligned}$$

2.3 Fourier Transform of a Convolution

Definition 2.2 *The convolution of f and g is the function $f * g$ defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x - t) dt.$$

Note also that

$$(f * g)(x) = \int_{-\infty}^{\infty} g(t) f(x - t) dt = (g * f)(x)$$

as can be shown by a change of variable.

Theorem: [Convolution theorem]

$$\mathcal{F}\left[(f * g)(x)\right] = \sqrt{2\pi} F(\mu) G(\mu).$$

Proof: Assignment

2.4 Fourier Sine and Cosine Transforms

The Fourier Cosine Transformation $\mathcal{F}_c[f(\cdot)]$ and Fourier Sine Transformation $\mathcal{F}_s[f(\cdot)]$ of $f : \mathbb{R} \to \mathbb{R}$ are defined as follows,

$$\begin{aligned}
\mathcal{F}_c[f(\cdot)] &= F_c(\mu) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \mu x \, dx \\
\mathcal{F}_s[f(\cdot)] &= F_s(\mu) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \mu x \, dx
\end{aligned}$$

Example

Find the Fourier Cosine and Sine transforms of the function $f(x) = e^{-ax}, \ a > 0, \ x > 0.$

Solution

$$
\begin{aligned}
F_c(\mu) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\mu x dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\mu x dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\left(\frac{e^{i\mu x}+e^{-i\mu x}}{2}\right)dx \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \left(e^{-(a-i\mu)x}+e^{-(a+i\mu)x}\right)dx \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[-\frac{1}{(a-i\mu)}e^{-(a-i\mu)x}-\frac{1}{(a+i\mu)}e^{-(a+i\mu)x}\right]\Bigg|_0^\infty \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{(a-i\mu)}+\frac{1}{(a+i\mu)}\right] \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{2a}{a^2+\mu^2}\right] \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+\mu^2}\right]
\end{aligned}
$$

Similarly, (complete the missing steps)

$$
\begin{aligned}
F_s(\mu) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\mu x dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin\mu x dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\left(\frac{e^{i\mu x}-e^{-i\mu x}}{2i}\right)dx \\
&= \\
&. \\
&. \\
&. \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{\mu}{a^2+\mu^2}\right]
\end{aligned}
$$

3 Application to Partial Differential Equations

Fourier transforms are used to convert linear partial differential equations into ordinary differential equations. This is possible because Fourier transforms convert derivatives into polynomial multiplication. The resultant ordinary differentiall equation can then be resolved by standard ODE techniques.

3.1 Heat Equation for an Infinite Rod

Consider the Initial Value Problem (IVP)

$$\begin{aligned} u_t(x,t) - ku_{xx} &= 0, \quad -\infty < x < \infty, \quad t > 0, \quad k > 0 \\ u(x,0) &= f(x) \end{aligned}$$

Let

$$\mathcal{F}[u(x,t)] = U(\mu,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\mu x} dx,$$

be the Fourier transform of $u(x,t)$ with respect to the x-variable, and let

$$\mathcal{F}^{-1}[U(\mu,t)] = u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\mu,t) e^{i\mu x} d\mu,$$

be its inversion formula.
Taking the Fourier transform of the PDE,

$$\mathcal{F}[u_t(x,t)] - k\mathcal{F}[u_{xx}(x,t)] = \mathcal{F}[0],$$

since $\mathcal{F}$ is a linear operator, and interchanging the time derivative with the integral, we obtain

$$\frac{\partial}{\partial t} U(\mu,t) + k\mu^2 U(\mu,t) = 0$$

The above equation for a fixed value μ is an ordinary differential equation in t which we solve to obtain

$$U(\mu,t) = A(\mu) e^{-k\mu^2 t}, \quad A(\mu) \text{ arbitary function.}$$

Taking Fourier transform of the initial condition and setting $t = 0$, we get

$$U(\mu,0) = A(\mu) = F(\mu)$$

where

$$F(\mu) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} \, dx$$

is the Fourier transform of the initial data $f(x)$. Therefore

$$U(\mu,t) = F(\mu) e^{-k\mu^2 t}$$

To determine $u(x,t)$, we use the inversion formula

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} U(\mu,t)e^{i\mu x}d\mu \\ &= \int_{-\infty}^{\infty} F(\mu)e^{-k\mu^2 t}e^{i\mu x}d\mu, \\ \text{where} \quad F(\mu) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\mu y}dy, \quad \text{thus} \\ u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-i\mu(y-x)}e^{-k\mu^2 t}dyd\mu. \end{aligned}$$

3.2 Half Range Problems

We now consider applications that involve the use of Fourier sine or cosine transforms

1. Fourier sine/cosine transforms can be used when the transformed variable is restricted to a semi-infinite interval.

2. The choice of the cosine-sine transform is dictated by the boundary condition

Recall

$$\begin{aligned} \mathcal{F}_c[f(x)''] &= -\sqrt{\frac{2}{\pi}}f'(0) - \mu^2 F_c(\mu) \\ \mathcal{F}_s[f(x)''] &= \sqrt{\frac{2}{\pi}}\mu f(0) - \mu^2 F_s(\mu) \end{aligned}$$

One can prove the following

$$\begin{aligned} \mathcal{F}_c[u_{xx}(x,t)] &= -\sqrt{\frac{2}{\pi}}u_x(0,t) - \mu^2 U_c(\mu) \\ \mathcal{F}_s[u_{xx}(x,t)] &= \sqrt{\frac{2}{\pi}}\mu U_s(0,t) - \mu^2 U_s(\mu) \end{aligned}$$

Now we not that

$$\mathcal{F}_c\left[\frac{\partial u(x,t)}{\partial t}\right] = \frac{\partial U_c(\mu,t)}{\partial t}$$

and

$$\mathcal{F}_s\left[\frac{\partial u(x,t)}{\partial t}\right] = \frac{\partial U_s(\mu,t)}{\partial t}$$

Remarks:

1. the Fourier cosine transform is useful when $u_x(0,t)$ is defined in the problem.

2. the Fourier sine transform is useful when $u(0,t)$ is defined in the problem.

Example

Consider the diffusion equation (heat equation)

$$\begin{aligned} u_t(x,t) - ku_{xx}(x,t) &= 0, \quad 0 < x < \infty,\ t > 0 \\ \text{I.C.} \qquad u(x,0) &= f(x), \quad 0 < x < \infty,\ t > 0 \\ \text{B.C.} \qquad u(0,t) &= g(t), \quad t > 0. \end{aligned}$$

Solution

We choose the Fourier sine transform in x since the temperature at $x = 0$, $u(0,t) = g(t)$ is provided. Taking the Fourier sine transform of the PDE with respect to the x variable, we obtain

$$\begin{aligned} \frac{\partial U_s(\mu,t)}{\partial t} - k\left[\sqrt{\frac{2}{\pi}}\mu u(0,t) - \mu^2 U_s(\mu)\right] &= 0 \\ \Rightarrow \quad \frac{\partial U_s(\mu,t)}{\partial t} + k\mu^2 U_s(\mu) - \sqrt{\frac{2}{\pi}}k\mu u(0,t) &= 0 \\ \Rightarrow \quad \frac{\partial U_s(\mu,t)}{\partial t} + k\mu^2 U_s(\mu) &= \sqrt{\frac{2}{\pi}}k\mu g(t) \end{aligned}$$

Solving the ODE above, we obtain

$$\begin{aligned} U_s(\mu,t) &= \sqrt{\frac{2}{\pi}}k\mu e^{-k\mu^2 t}\int g(t)e^{k\mu^2 t}dt + A(\mu)e^{-k\mu^2 t} \\ &= \sqrt{\frac{2}{\pi}}k\mu \int_0^t g(\tau)e^{-k\mu^2(t-\tau)}\, d\tau + A(\mu)e^{-k\mu^2 t} \end{aligned}$$

setting $t = 0$, we obtain
$U_s(\mu,0) = A(\mu) = F_s(\mu)$

$$\Rightarrow \quad U_s(\mu,t) = F_s(\mu)e^{-k\mu^2 t} + \sqrt{\frac{2}{\pi}}k\mu \int_0^t g(\tau)e^{-k\mu^2(t-\tau)}\, d\tau$$

and the solution is the inverse sine transform of $U_s(\mu,t)$. The inversion formula of $U_s(\mu,t)$ yields the solution $u(x,t)$ as

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty U_s(\mu,t)\sin\mu x\, d\mu \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\mu)e^{-k\mu^2 t}\sin\mu x\, d\mu + \frac{2}{\pi}k\int_0^\infty\int_0^t \mu g(\tau)e^{-k\mu^2(t-\tau)}\sin\mu x\ \ d\mu\, d\tau \\ &= \frac{2}{\pi}\int_0^\infty\int_0^\infty f(s)e^{-k\mu^2 t}\sin\mu s\ \sin\mu x\, d\mu\, ds + \frac{2}{\pi}k\int_0^t\int_0^\infty \mu g(\tau)e^{-k\mu^2(t-\tau)}\sin\mu x\, d\mu\, d\tau, \end{aligned}$$

which is the integral representation of the solution $u(x,t)$ in terms of the data $u(x,0) = f(x)$ and $u(0,t) = g(t)$.