UNIVERSITY OF ZIMBABWE

VALUATION OF AMERICAN TYPE OPTIONS USING NUMERICAL APPROXIMATIONS

by

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To my mother, Mrs Maramba, who has always been my guiding light and unwavering source of support throughout my academic journey. Her unconditional love, sacrifice, and belief in me has been the driving force behind my success. This thesis is dedicated to her with all my love and gratitude

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Abstract

This study presents an examination of three finite difference methods for pricing American options. The methods considered are the forward difference (explicit), backward difference (implicit), and Crank-Nicolson schemes. We apply these methods to solve the transformed Black-Scholes partial differential equations and price American options under different scenarios.

We examine the effectiveness of the methods in capturing the free boundary problem and examine their accuracy and efficiency in pricing options. We also examine the impact of different parameters such as volatility and interest rate on option prices and demonstrate how the methods perform under varying conditions.

The results show that the Crank-Nicolson scheme provides the most accurate option prices, while the explicit scheme is the fastest method. The findings of this study provide important insights for investors and financial analysts in selecting appropriate numerical methods for valuing American options.

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Chapter 1

Introduction

The significance of financial derivatives has grown in the realm of finance. These are financial instruments whose value is based on or derived from other underlying variables. For instance, a stock option is a derivative whose value is linked to the stock price. The emergence of derivative markets in the modern economy has led to the trading of various options by investors. The research paper will center on American Options, which are a specific type of options.

1.1 Background

Options are financial contracts that allow the holder to buy (Call Option) or sell (Put Option) an asset at an agreed-upon price in the future, without the obligation to do so. Call Options gives the holder the right but not obligation to buy an underlying asset whereas Put Options gives the holder the right but not obligation to sell an underlying asset. Options can be obtained through purchase, compensation, or as part of a complex financial deal, and they are considered assets that have a value based on various factors such as the underlying asset price, time until expiration, market volatility, risk-free interest rate, and strike price. The financial markets offer a variety of option contracts, including European options, American options, Asian options and many more. The names of these options are not regional or based on their origin, but rather describe different styles of exercising options. A European option holder can only exercise it on the expiration date, whereas an American option holder can exercise the contract at any time before the expiration date. An Asian option is distinct in that it is priced using the underlying asset's prior price fluctuations.

Options are a commonly employed financial tool by investors for purposes of both hedging and speculation, because of this, option valuation is a critical topic in financial engineering. Hedgers use options to reduce risk by buying or selling commodities at favorable prices. Speculators uses options to gain leverage and increase profits.

European options are easy to value since the exercise boundary is fixed on the expiration date, their value can be easily calculated using the famous Black-Scholes (1973) model. American options, on the other hand, have a built-in opportunity for traders to receive a larger reward due to their flexible execution window. As a result, they control the majority of the major financial markets' option contracts. However, because of their early-exercise characteristic, the Black–Scholes model results in a free boundary value, as a result closed form solutions cannot be found and formulas for the exact solutions are too complex to be practically usable. Thus for this reason, it is essential to find a good approximation to the option price, the major research on this paper will focus on presenting efficient and accurate numerical algorithms for valuation of American options.

1.1.1 European Options

The **European Call option** is a financial contract that outlines an agreement between two parties, specifying the conditions for the option holder to purchase a predetermined asset, known as the underlying, on a specific future date, called the expiration date, at a predetermined price, known as the exercise or strike price. It is important to note that the holder of the Call option has the right but not the obligation to exercise the contract.

Valuation is a crucial aspect of options trading, and the European call option's value is determined by several factors. These factors include the exercise price, represented by E, and the underlying asset price at the expiration date, represented by $S(T)$. If the underlying asset price at expiry, $S(T)$, is greater than the exercise price, E, the call option holder can buy the asset for E and sell it immediately in the market for $S(T)$, resulting in a profit of $S(T)$ - E. However, if E is greater than $S(T)$, the option holder does not gain anything. Thus, the value of a European call option can be calculated using a formula that takes into account these factors which is given below.

$$C(S,T) = \max\left(S(T) - E, 0\right) \tag{1.1.1}$$

A **European Put Option** is a financial contract that specifies the terms for the option holder to sell a predetermined asset, referred to as the underlying, at a predetermined exercise or strike price on a specific future date, known as the expiration date. It is important to note that this contract is only valid until the expiration date and will expire thereafter.

A European Put Option is valued based on the exercise price, represented by E, and the underlying asset price at the expiration date, represented by $S(T)$. If the asset price at expiry, $S(T)$, falls below the exercise price, E, the option holder can sell the asset at the predetermined price, E, and immediately purchase it in the market for $S(T)$, resulting in a profit of E-$S(T)$. However, if E is less than $S(T)$, the option holder does not make a profit. Therefore, the value of a European Put Option can be calculated using a formula that considers these factors given as.

$$P(S,T) = \max(E - S(T), 0) \tag{1.1.2}$$

Valuing European options is simple since the exercise border is defined at the expiry date, represented as T. Analytic solutions for non-dividend-paying European options can be derived using the option pricing models proposed by Black-Scholes (1973).

1.1.2 American Options

An American Option is a financial instrument that grants the holder the right, but not the obligation, to buy (Call Option) or sell (Put Option) a specified asset from the writer at a predetermined price (E) at any time between the start date and a specified expiration date (T) in the future. Unlike European Options, American options allow the holder to exercise the option at any time before the expiration date, providing additional rights that can potentially increase their value. However, the absence of a fixed exercise boundary in American options makes it challenging to determine the optimal time to exercise the option, resulting in a free boundary problem that requires careful modeling of the option price. These factors form the basis of research on American options.

1.2 The Statement of the Problem

The free boundary issues are extremely non-linear models because the free boundaries are unknown in advance and must be specified as part of the solution. For the reasons described above, and since solving these practical difficulties is frequently exceedingly difficult, studies on these topics have continually drawn the interest of researchers in all aspects. The American option pricing model belongs to the class of free boundary problems, which are exhausting and difficult to solve. Although several effective numerical approaches can achieve some solutions, it is well known that it is difficult to manage the precision and stability, as they are substantially impacted by the singularity at the exercise boundary approaching expiration. The front-tracking technique and the front-fixing method are the two basic strategies for dealing with the free boundary challenges. The front-tracking techniques track the free boundary's location over time. The Landau transformation is used in the front-fixing approach to immobilize the ideal stopping border. Accurate valuation of American options is a tough problem due to the potential of early exercise, and the success of a numerical method for addressing the American problem is heavily reliant on its ability to precisely pinpoint the **optimal exercise border**. In this research, we present some fresh insights on the performance of the suggested strategy in order to carefully and precisely solve the obstacles mentioned above on the path to enhance accuracy in the American option problem.

1.3 Aim

The aim of this dissertation is to determine a reliable technique for American options valuation.

1.4 Objectives

The following are the objectives of this dissertation.

- To develop a precise method that can determine the position of the free boundary and also calculate the corresponding option valuation at discrete time intervals leading up to the expiry date.

- To transform the Black-Scholes equation into a parabolic equation without dimensions, and subsequently applying numerical methods to discretise the problem and solve for a solution.

Chapter 2

Literature Review

In 1973, Merton stated valuation of the American options as a free boundary problem via the Black–Scholes partial differential equation framework. Then, several analytical and numerical approaches were developed to solve the American options as a free boundary problem. For an analytical approach, Geske and Johnson (1984) presented a valuation formula for the American put option using the Richardson extrapolation approximation. This formula evaluation process yields a polynomial expression that is identical to the polynomial expression used to evaluate the integral terms in the Black-Scholes for European put option formula. For pricing American put options, P. Carr, R. Jarrow, and R. Myneni introduced a unique quasi-analytical approximation method in 1992. By dividing the option value into the European put price and the early exercise premium, this approach offers a number of advantages, including better understanding, more effective numerical evaluation, tighter analytical bounds, and the potential for creating new analytical approximations. MacMillan (1986) and Barone-Adesi and Whaley (1987) made significant advancements in the field of American option pricing by devising analytical approximations that are widely acknowledged. They adopted the quadratic approximation method in determining the American Put and American call Value. S. P Zhu (2006) proposed a mathematical formula that utilizes analytical approximations to estimate the optimal exercise price of the American put option using the Laplace transformation. A Laplace transform approximation of the moving boundary conditions was used to solve the Black-Scholes equation in Laplace space and arrive at the formula. The resultant equation showed that the ideal exercise price, which is a major obstacle in pricing American options, is made up of the perpetual optimal exercise price plus a declining early exercise premium dependent on the amount of time left before option expiry. Additionally, a mathematical formula was developed to determine the cost of American put options. Similar to the cumulative distribution function of the common normal distribution used in the pricing of European options, this formula indicates a time-independent perpetual upper limit, lowered by an early exercise cost (a negative premium) that may be stated by a straightforward integration.

Due to complications in obtaining the analytical solution of these problems, various numerical algorithms have been proposed by many researchers. Firstly, Brennan and Schwartz (1977) and Schwartz (1978) used explicit and implicit finite differences for solving the American option problems. After those research, other writers offered a variety of finite difference-based solution techniques for the option dynamic pricing. Muthuraman (2006) presented a shifting boundary approach for generating the optimal exercise price at each time point in an approximate manner. The technique is essentially iterative and entails establishing an initial boundary that is less expensive than the optimal exercise price, then revising the boundary in accordance with the no-arbitrage principle. It has been demonstrated that using this strategy yields convergent updated limits that monotonically approach the boundary of the ideal exercise, turning the moving boundary problem into a string of fixed boundary issues. This approach has the capacity to greatly cut computation time since it only needs a small number of iterations to achieve the necessary level of convergence.

2.1 The Factors Influencing the Valuation of Options

Stock option pricing is influenced by six key factors:

1. **The prevailing stock price (S(0))and the strike price (E)**. In the case of a future call option execution, the payment received is contingent on the differential between the stock price and the strike price. Consequently, call options become increasingly attractive as the stock price rises, but less desirable as the strike price escalates. Conversely, put options operate in an opposite manner to call options, as their payout is predicated on the disparity between the strike price and the stock price. Thus, put options decrease in value when the stock price increases, but appreciate in value as the strike price rises.

2. **The time to expiration *(T)***. Consider the following impact of the expiration date. As the time to expiry approaches, both put and call American options

become more valued. Consider only two alternatives in terms of the expiration date. The possessor of a long-dated option benefits from a wider range of exercise possibilities compared to the owner of a short-dated option. Consequently, the long-dated option should always be valued at a minimum equal to the short-dated option

3. **The variability of the stock price**. Volatility of stock price reflects the uncertainty about future stock price fluctuations. When volatility rises, so is the likelihood that the stock may perform well or poorly. For shareholders, these two scenarios typically offset each other, resulting in little net effect. However, this is not the case for call or put option holders. Call possessors benefit from price increases while having limited downside risk in case of a price drop, as their maximum loss is equivalent to the option's price. Conversely, put possessors profit from price declines while having no risk if prices rise. In times of increased volatility, the value of both calls and puts tends to appreciate.

4. **Free interest rate, *r***. The risk-free interest rate influences option pricing by raising investors' projected stock returns while diminishing the present value of any earnings received by the option owner. This causes put option values to fall while call option values to rise. In actuality, however, a rise or drop in interest rates may have an influence on stock prices, resulting in a net effect that might reduce the value of a call option while increasing the value of a put option amid an interest rate increase and corresponding stock price fall. A fall in interest rates and an increase in stock prices, on the other hand, might enhance the value of a call option while decreasing the value of a put option.

5. **The expected dividends over the course of the option's life**. Dividends can cause a drop in the stock price on the ex-dividend date, which has ramifications for option values. Call option values, in particular, tend to fall while put option values rise. As a result, the value of a call option is negatively related to the amount of expected dividends, but the value of a put option is positively related to the size of projected payouts.

2.2 Model for Stock Prices

To accurately price options, we need to establish an analytical framework that depicts the behaviour of the underlying stock. The stock price reflects the level of investor confidence, which is subject to random fluctuations according to the efficient market hypothesis. In other words, this theory posits that there is no systematic pattern or predictability in the movements of stock prices that is :

- The current price fully incorporates all past information, but it does not take into account any additional information beyond that.
- The market efficiently responds to any fresh information related to the stock.

Assuming the two conditions mentioned above, stock prices are regarded to be governed by a Markov process. **A Markov process** is a stochastic process where the probability distribution of the next state depends solely on the current state, and not on any previous states. In other words, the future behaviour of the process is independent of its history, given the current state.

Let S be the stock price at time t, and consider a subsequent time interval dt during which S changes to $S + dS$. We can decompose the resulting return into two components: a deterministic component that resembles the return on a risk-free investment, and a stochastic component that captures the variability in the stock's return. The deterministic component contributes $\frac{dS}{S}$ to the overall return and is given by

$$\mu dt \tag{2.2.1}$$

The parameter μ in this context refers to the average upward movement in stock prices, which is sometimes called the drift.

The second component that contributes to $\frac{dS}{S}$ reflects the random change in the stock price due to external factors. This component is modelled as a random sample drawn from a normal distribution with a mean of zero, which captures the stochastic nature of the stock's behaviour. The contribution of this component is given by:

$$\sigma dX \tag{2.2.2}$$

σ represents the degree of variability in the stock's returns, as measured by its volatility.

Thus combining expressions 2.2.1 and 2.2.2 contributions we get

$$\frac{dS}{S} = \sigma dX + \mu dt \tag{2.2.3}$$

dX is a term used to describe the stochastic element of asset prices, which is a fundamental component of the Wiener process, the basic mathematical model for generating stock prices and has the following properties.

- dX follows a normal distribution,
- The expected value of dX is zero, while its variance is equal to dt.

2.3 Concept of Arbitrage

Arbitrage is a financial strategy that involves exploiting pricing discrepancies between two or more markets. This is done by executing a combination of matching deals that capitalize on the imbalance. The outcome is a profit that is commensurate with the difference in market prices. Arbitrage transactions involve no cash inflows or outflows in any probabilistic or temporal state, except for one state that generates a positive cash flow. Essentially, arbitrage presents the opportunity for a risk-free profit after transaction costs. Arbitrage in the context of American options refers to the process of making a free of risk gain by taking advantage of price discrepancies between different markets for the same underlying asset. In the case of American options, which permit the exercise of the option by the holder at any time prior to the option's expiration date, arbitrage opportunities can arise when the price of the option in one market is mispriced relative to the price of the same option in another market. Traders can buy the underpriced option in one market and simultaneously sell the overpriced option in another market, locking in a profit without taking on any risk. However, arbitrage opportunities in financial markets are typically short-lived, as market participants quickly move to take advantage of them, which tends to correct the price discrepancies. There are several several types of arbitrages, the following are some of arbitrage opportunities that may occur:

1. **Conversion arbitrage**: This type of arbitrage involves buying a call option, selling a put option, and buying the underlying asset at the same time. If the option prices are not consistent with the underlying asset price, an arbitrage opportunity may exist.

2. **Reversal arbitrage**: This type of arbitrage involves buying a put option, selling a call option, and selling the underlying asset at once. If the option prices are not consistent with the underlying asset price, an arbitrage opportunity may exist.

3. **Box spread arbitrage:** This type of arbitrage involves buying a bull call spread (A call option with a reduced strike price and a call option with an elevated strike price) and selling a bear put spread (A put option with a reduced strike price and a put option with an elevated strike price) at the same time. If the option prices are not consistent with the underlying asset price, an arbitrage opportunity may exist.

4. **Dividend arbitrage:** This type of arbitrage involves buying a stock that pays a dividend and selling a call option on that stock. If the option prices are not consistent with the dividend payments, an arbitrage opportunity may exist.

5. **Early exercise arbitrage:** This type of arbitrage involves exercising an American option early when it is profitable to do so. If the option holder has perfect information and can exercise the option at any time, an arbitrage opportunity may exist.

Arbitrage is an important concept in finance because it helps to ensure that prices remain efficient and that market participants have access to the same information. By taking advantage of price discrepancies, arbitrageurs help to align prices and reduce the chances of market inefficiencies.

2.4 The Black Scholes Model

The Black-Scholes formula constitutes the initial stage of option pricing where adaptations of the valuation of American options will be based on. The derivation of this partial differential equation is based on the risk neutral valuation principle which can be summarized as follows

- The writer of a call hedges his position or exposure by holding a number of units of the underlying asset in order to create a riskless portfolio.
- When an efficient market exists with no chance of arbitrage, a portfolio that is free of risk must produce a return that corresponds to the risk-free interest rate.

2.4.1 Assumptions

The following assumptions are adopted in the derivation of the Black Scholes Pricing Model

1. Trading occurs continually throughout time.
2. The risk-free interest rate (r) remains constant and traceable throughout the option's lifetime
3. There are no dividend disbursements from the underlying asset.
4. The market is devoid of all frictions, including taxes and commissions.
5. The assets are infinitely divisible.
6. Short selling is permissible.
7. The market is free from arbitrage opportunities.

Now to derive the formula, three assets are said to be available in the market and these are

1. riskless bond/bank account whose unit value $A(t) = A_t$ at time tis driven by the equation

$$dA(t) = rA(t)dt$$

2. Risky asset/stock whose unit price $S(t) = S_t = S$ at time t satisfies the equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where μ and σ are constants and B_t is a 1-dimensional Brownian Motion.

3. European call option written over the stock price $C = C(t, S_t)$

Consider a portfolio which involves writing a European call option and buying Δ units of the stock. The value $\Pi(t) = \Pi$ of the portfolio at time t is given by

$$\Pi(t) = C - \Delta S_t \tag{2.4.4}$$

Since both $\Pi(t)$ and Π are Ito processes they can be differentiated as follows.

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2$$

$$= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\mu S_t dt + \sigma S_t dB_t)^2$$

$$= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 dt$$

$$= \left(\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2\right) dt + (\sigma S_t)\frac{\partial C}{\partial S}dB_t \tag{2.4.5}$$

Now since the $\Pi = C - \Delta S$

$$d\Pi = dC - \Delta dS$$

$$= \frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2)dt + (\sigma S_t)\frac{\partial C}{\partial S}dB_t - \Delta(\mu S_t dt + \sigma S_t dBt)$$

$$= (\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 - \Delta \mu S_t)dt + (\sigma S_t \frac{\partial C}{\partial S} - \Delta \sigma S_t)dB_t$$

$$= (\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2) + (\frac{\partial C}{\partial S} - \Delta)\mu S_t)dt + (\frac{\partial C}{\partial S} - \Delta)\sigma S_t dB_t \tag{2.4.6}$$

The idea behind the Black Scholes analysis is to construct a riskless portfolio in the underlying asset and the option. In order to achieve this we choose $\Delta = \frac{\partial C}{\partial S}$ and by doing so

$$d\Pi = (\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2)dt \tag{2.4.7}$$

Now since $\Pi(t)$ is a riskless portfolio it must earn riskless interest since they are no arbitrages in the economy. It follows that $\Pi(t)$ satisfies the equation

$$d\Pi = r\Pi(t)dt$$

$$= r(c - \Delta S)dt$$

$$= (rC - rS\frac{\partial C}{\partial S})dt \tag{2.4.8}$$

Now comparing the two equations (2.49) and (2.4.10)we get

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 = rC - rS\frac{\partial C}{\partial S}$$

which can be rearranged to

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 + rS\frac{\partial C}{\partial S} - rC = 0 \tag{2.4.9}$$

which Black-Scholes partial differential equation and describes how European call options react to changes in the underlying asset price and time. As a result, any derivative security that is paid in advance and whose price is decided purely by S and t must follow the equation.

2.4.2 Boundary Conditions

The partial differential equation that was generated above is a backward parabolic equation. As a result, boundary conditions must be set in order to guarantee a unique outcome. On S and t, two requirements are placed. Since the formulae apply to European options, we begin by examining the boundary conditions for a Call option. A European call option's payout value is known with certainty at $t = T$:

$$C(S, T) = \max(S - E, 0)$$

The aforementioned condition is the ultimate requirement for the partial differential equation when evaluating the call option. If $S = 0$ at the expiry date, the payoff is zero, which implies that the option is valueless at $S = 0$ even if the time to expiration is long. Therefore, at $S = 0$, we obtain:

$$C(0, t) = 0$$

As the stock price increases, the probability of it being exercised rises, and the significance of the exercise's magnitude declines. Hence, as $S \to \infty$, the option's value approaches that of the underlying asset, and we represent it as follows:

$$C(S, t) \sim S \text{ as } S \to \infty$$

Thus for options without possibility of early exercise the above conditions can be solved to give the Black-Scholes value for call options.
Similarly for put option final condition is the payoff:

$$P(S, T) = \max(E - S, 0)$$

If S equals zero, it must remain at zero. Under these circumstances, the final payoff for the put is a fixed value of E. To determine $P(0, t)$ we compute the present value of E received at time T. Assuming constant interest rates, the boundary condition at $S = 0$ is:

$$P(0, t) = Ee^{-r(T-t)}$$

As S$\to \infty$ the option is unlikely to be exercised and so

$$P(S, t) \to 0 \text{ as } S \to \infty$$

2.4.3 The Extensions of the Model

The preceding section concluded the Black-Scholes analysis of the fundamental European call and put options. Although the derived formula may be helpful, it may not be sufficient for more complex scenarios. The model assumed that no dividends are distributed. However, when assets pay dividends to their shareholders, the option price of an underlying asset is affected by these payments, as discussed in section 2.1. Therefore, an extension is necessary for the Black-Scholes partial differential equation to incorporate dividends. When taking dividends into account, the following questions are addressed:

- What is the timing and frequency of dividend disbursements?
- What is the magnitude of the dividend payments?

Either a deterministic or stochastic model can be used to describe the dividend payment amount. In this project, we will only take into account the stocks that pay dividends and whose time and amount are known at the beginning of the option life.

Let's say the underlying asset distributes a dividend of D_oSdt over a time period dt, where D_o is a constant amount. The stock price S and not time determines whether the dividend is paid. The dividend yield, or percentage of asset price paid out per unit of time, represented by the dividend D_oSdt is constant and continuous.

It can be demonstrated that, when arbitrage is taken into account, in addition to the typical swings, the asset price in each time-step dt must fall by the amount of the dividend payout. The previously described random walk is now changed to become

$$ds = \sigma SdX + (\mu - D_o)Sdt$$

Taking into account the impact of dividend payments to the hedged portfolio, we get an amount (D_oSdt) for each asset held, and because we own ($-\Delta$) of the underlying, the portfolio changes by an amount.

$$-D_oS\Delta dt$$

Thus

$$d\Pi = dC - \Delta dS - D_o\Delta dt$$

Thus using the same analysis from the previous section we obtain

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 + (r - D_o)S\frac{\partial C}{\partial S} - rC = 0 \tag{2.4.10}$$

which is the Black Scholes formula with an extension of dividends. The only modification made to the boundary conditions is

$$C(S,t) \sim Se^{-D_o(T-t)} \text{ as } S \to \infty$$

This occurs because as S approaches infinity, the option's value converges to that of the asset price, without considering dividends.

Chapter 3

The Model and Transformation

In Chapter 1, subsection 1.1.2 it was discussed that American options have an additional that the option may be exercised at any time throughout the term of the option. Thus the formula derived in section 2.3 do not necessarily work for American type options. This thesis serves two purposes. The first goal is to create a price formula exclusively for American-style options, and the second goal is to look at the use of numerical techniques to value these choices.

3.1 Valuation Formula for American Options

There is a sizable range of underlying asset values S where the European put option's value is smaller than its corresponding reward function while an American option is still in effect prior to expiry. If the option is exercised and S is within this range, such that $P(S,t) < max(E-S,0)$, an arbitrage opportunity exists. We may execute the option by selling the asset for E and making a risk-free profit of $E-P-S$ by first acquiring the asset on the market for S and the option at the same time for P. This advantage is transient, though, since arbitrageurs' demand would drive up the option's value. The early exercise of American options is therefore subject to certain limitations, we must infer.

$$V(S,t) \geqslant max(S-E,0)$$

In certain scenarios involving American options, it may be optimal for the option holder to exercise their option at specific values of S. This is the case which differentiate American options from the European options. As a result the Black Scholes model fails to hold for the values of S on American options. In the case of American options, it is required to ascertain not only their value but also to make a choice on whether or not to exercise them for each distinct underlying asset value S.

We can use the notation $S_f(t)$ to refer to the exercise price that yields the highest value for the option. As mentioned earlier, the value of $S_f(t)$ is not known in advance, and we are also uncertain about where to apply the boundary conditions. Hence, this problem is referred to as the **free boundary problem**. Thus valuation of American options involves solving a free boundary problem which can be viewed as the classical **obstacle problem**. The free boundary represents the optimal exercise boundary, or a specific point at which it is most advantageous to exercise the option. The obstacle problem arises because the option holder bears the responsibility of making the decision when to exercise the option, the optimal exercise time and price depend on the underlying asset price and other market conditions.

3.1.1 The Obstacle Problem

Friedman (1982) and Wilmott (1995) described the free boundary problem as a classic obstacle problem, which can be illustrated using the example of an elastic thread crossing a smooth object fixed at positions A and B. The displacement of the string is represented by $u(x)$ and the height of the object by $f(x)$, where the endpoints of the string are represented by $x = \pm 1$. The challenge is to determine whether the string lies above or on top of the object, while ensuring that the string and its slope are continuous. However, the exact point where the string and the object make contact is unknown. The free boundary problem involves identifying the contact region defined by the points $P(x = x_P)$ and $Q(x = x_Q)$. When the string is in contact with the object, the contact region is characterized by a concave downward shape, where $u = f$ and $u'' < 0$. On the other hand, when the string is above the object, $u > f$ and $u'' = 0$. Moreover, it is assumed that.

$$f(\pm) < 0, f(x) > 0, \text{for some}, -1 < x < 1, f'' < 0 \tag{3.1.1}$$

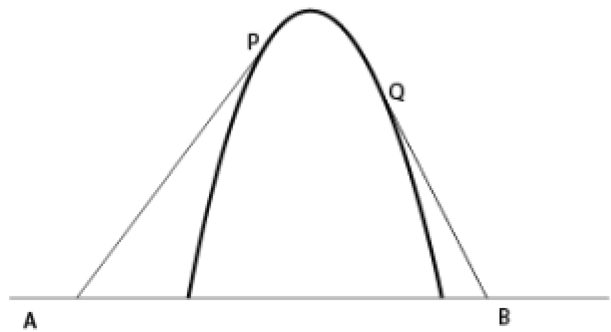
h 

Figure 3.1: The Classical Obstacle Problem.

To ensure that there is only one contact region, the obstacle problem requires the identification of $u(x)$ and the points P and Q such that:

$$\begin{aligned} u(-1) = 0, u(1) = 0 \\ u'' = 0, -1 < x < x_P, x_Q < x < 1 \\ u(x_P) = f(x_P), u'(x_P) = f'(x_P) \\ u(x_P) = f(x_Q), u'(x_Q) = f'(x_Q) \\ u(x) = f(x), x_P < x < x_Q \end{aligned} \tag{3.1.2}$$

A possible approach for solving this problem is to restate it as a linear complementary problem, given by:

$$u'' \cdot (u - f) = 0, -u'' \geq 0, u - f \geq 0 \tag{3.1.3}$$

The constraints of the issue imply that $u(-1) = u(1) = 0$ and that u and u' are continuous. The free boundary points are not explicitly given in this formulation, but are integrated implicitly via the restriction $u \geq f$. To address the resultant issue, a variety of numerical approaches might be used.

It can be demonstrated that American options are distinctively defined by a collection of limitations that share many similarities with those of the obstacle problem. The restrictions for American options consist of:

- ensure that the value of the option is either equal to or greater than its corresponding payoff function

- Instead of using the Black-Scholes equation, an inequality can be utilized.
- The option value to exhibit continuity as a function of the underlying asset value S.
- Sustaining the continuity of the option delta (i.e., its slope) is crucial.

Given that there are two fundamental forms of options, namely the Put option and the Call option, we will initially investigate the scenario of a Put option, followed by the examination of the Call option.

3.1.2 American Put Option

Lets examine an American put option characterized by a value of $P(S,t)$. As previously demonstrated, the exercise boundary for this option exists when $S = S_f(t)$, which indicates that the option should be exercised if S is less than $S_f(t)$ and retained otherwise. If we assume that $S_f(t)$ is less than E and we have the payoff function defined as follows:

$$P(S,T) = \begin{cases} E - S & \text{if } E - S > 0 \\ 0 & \text{if } E - S \leq 0 \end{cases} \tag{3.1.4}$$

The gradient of the payoff function at the point of contact is -1. This is because since $S_f(t)$ represents the exercise boundary for the American put option, we know that $S_f(t) < E$. Therefore, the maximum function (payoff function) evaluates to $E - S$ when $S = S_f(t)$, and the gradient of this function at the point of contact is simply the derivative of $E - S$ with respect to S, which is -1. Hence, the gradient of the maximum function of $E - S$ is negative one at the point of contact.

The Black Scholes partial differential equation is derived from an arbitrage argument, which is only partially applicable to American options. However, the relationship between arbitrage and the Black Scholes operator remains closely linked. The formula now produces an inequality instead of an equation. The delta hedged portfolio is established in the same manner as before, with the same delta selection. However, in the case of American-style options, it may not be feasible to hold the option both long and short since there are instances when it is advantageous to exercise the option. This implies that the option writer may be subject to exercise. Unlike European options, American-style options do not yield a unique value for the portfolio's return using the arbitrage argument. Instead, it results in an inequality, which indicates that the portfolio's return cannot exceed that of the bank deposit. For American put options, this inequality takes the following form:

$$\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0 \tag{3.1.5}$$

To summarize the American put option is expressed as a free boundary problem, with $0 \leq S < S_f(t)$ being the optimal exercise region where,

$$P = E - S,\ \frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0 \tag{3.1.6}$$

and $S_f(t) < S < \infty$ being the early exercise suboptimal region where.

$$P > E - S,\ \frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \tag{3.1.7}$$

The boundary conditions for $S = S_f(t)$ are that P and its slope are continuous:

$$P(S_f(t), t) = \max\,(E - S_f(t), 0) \tag{3.1.8}$$

$$\frac{\partial P}{\partial S}(S_f(t), t) = -1 \tag{3.1.9}$$

We see them as one boundary condition for determining the option value on the free boundary and another for determining the location of the free boundary. We need an extra condition to determine $S_f(t)$ because we don't know where it is a priori. Arbitrage considerations show that P's gradient should be continuous, which provides us with the additional condition we require.

3.1.3 American Call Option

The following discussion pertains to certain analytical aspects of the American call option. Recall that the value C which is a function of S and t of the call satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 + rS\frac{\partial C}{\partial S} - rC = 0 \tag{3.1.10}$$

when the exercise is not optimal . The payoff condition is

$$C(S,t) = \begin{cases} S - E & \text{if } S - E > 0 \\ 0 & \text{if } S - E \leq 0 \end{cases}$$

or

$$C(S,t) = \max\,(S - E, 0)$$

Since the option is exercisable at any point in time, we invariably have.

$$C(S,t) \geq \max\,(S - E, 0)$$

In the event that an optimal exercise boundary exists, denoted by $S = S_f(t)$, the following holds true when $S = S_f(t)$.

$$C(S_f(t), t) = S_f(t) - E \tag{3.1.11}$$

$$\frac{\partial C}{\partial S}(S_f(t,t)) = 1 \tag{3.1.12}$$

If an optimal exercise boundary represented by $S = S_f(t)$ exists, then the inequality $C(S,t) > \max(S-E, 0)$ must hold true. This is because a straightforward calculation reveals that $\max(S-E, 0)$ does not satisfy the Black-Scholes equation. As in the put option it can be replaced by the inequality.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 + rS\frac{\partial C}{\partial S} - rC \leq 0 \tag{3.1.13}$$

equality only holds when $C >\ \max\,(S - E, 0)$. As in the case of the American put , the financial reason for this is that early exercise is optimal. The rationale for this is that retaining the option would result in a lesser value than exercising it immediately and depositing the monies in a bank.

3.2 Model Transformation

To obtain the solutions obtained via numerical techniques for the free boundary problem posed by American options, we will endeavour to identify a transformation that simplifies the problem to a fixed boundary problem, from which the free boundary can be deduced later. In this thesis, we will examine the transformation method that utilizes a linear complementarity formulation.

3.2.1 Linear Complementarity Formulation

In the context of solving free boundary problems posed by American options, the linear complementarity formulation can simplify the problem by transforming it into a fixed boundary problem. This is achieved by introducing a complementary slackness condition, which ensures that the solution satisfies both the original problem and the transformed problem simultaneously. By using this approach, the free boundary can be deduced from the solution of the fixed boundary problem, which is much easier to solve numerically. In the case of American-style options, the complementary slackness condition takes the form of a variational inequality, which can be solved using numerical methods. Once the variational inequality is solved, the optimal exercise boundary can be deduced from the solution of the fixed boundary problem. Thus the solution to the variational inequality provides optimal stopping boundary for the American-style option, which is the moving boundary that we are interested in.

In this study, we extend the comparison between the obstacle problem and the Black-Scholes formulation of the free boundary problem for American options by demonstrating that the latter can also be simplified to a linear complementarity problem. To achieve this, we first convert the American put problem from its original (S, t) variables to (x, r), where we substitute $S = S_f(t)$ for $x = x_f(r)$. It is worth noting that since $S_f(t) < E$, $x_f(r) < 0$. The transformation is given by:

$$x = \frac{2}{k+1}\log\left(\frac{S}{E}\right) - \frac{k-1}{k+1}\sigma\sqrt{t}$$

$$r = \frac{1}{\sigma^2}\left(r_f - q + \frac{1}{2}\sigma^2\left(\frac{k-1}{k+1}\right)^2 t\right) \tag{3.2.14}$$

Following a change of variables involving the exercise price (E), risk-free rate (r_f), dividend yield (q), and volatility of the underlying asset (σ), a constant k is introduced and the resulting transformed payoff function is expressed in a specific form.

$$P(S,T) = g(x,r) = \max\left(E - e^{kx}, 0\right) = e^{-kr} \max\left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right) \tag{3.2.15}$$

This expression can be further simplified using the transformation for x and r given above, which yields:

$$\begin{aligned} g(x,r) &= e^{\frac{1}{2}(k+1)^2 r} \max\left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right) \\ &= e^{\frac{1}{2}(k+1)^2 r} \max\left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right) \end{aligned} \tag{3.2.16}$$

This expression is equivalent to the obstacle function $g(x,r)$ that appears in the linear complementarity problem for the American put option. Specifically, we can write:

$$g(x,r) = e^{\frac{1}{2}(k+1)^2 r} \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0)$$

To put it differently, the obstacle function assumes a value equivalent to the disparity between the exercise price and the converted asset price, but only if the converted asset price does not exceed zero. When the converted asset price is greater than zero, the obstacle function assumes a value of zero. This series of transformations enables the Black-Scholes equation to be modified as:

$$\begin{gathered} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial x^2} \text{ for } x > x_f(r) \\ u(x,r) = g(x,r) \text{ for } x \leq x_f(r) \end{gathered} \tag{3.2.17}$$

with the initial conditions

$$u(x,o) = g(x,r) = max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0)$$

and asymptotic behaviour

$$\lim_{x\to\infty} u(x,r) = 0$$

As $x \to -\infty$. We are in the region where early exercise is optimal and so $u = g$. we also have the crucial constraint

$$u(x,r) \geq e^{\frac{1}{2}(k+1)^2 r}\ max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0)$$

and the conditions that u and $\frac{\partial u}{\partial x}$ are continuous at $x = x_f(t)$ of of which follow from the corresponding conditions in the original problem. Given that we must confine every numerical method within a finite mesh, it would be prudent to also confine the problem within a finite interval. In essence, limiting the problem domain to a finite interval aligns with the restrictions imposed by the finite mesh of the numerical approach. This implies that the problem is being examined within the interval of $-x^- < x < x^+$, where x^- and x^+ are sufficiently large. As a result, the boundary condition is being enforced by the problem domain.

$$u(x^+, r) = 0 \;\; u(-x^-, r) = g(-x^-, r)$$

Within the context of finance, it is common to approximate exact boundary conditions by substituting small values of S and $P = E - S$ with larger values. This approximation typically leads to the expression $P = 0$

The obstacle and the American put problem satisfy constraints , suggests that the latter might also have the a linear complementarity formulation. The option problem is very similar to the obstacle problem, but with an obstacle which is time dependent , that is the transformed payoff function $g(x, r)$ we can now write

$$\left(\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial x^2}\right) \cdot (u(x,r) - g(x,r)) = 0$$

$$\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial x^2} \geq 0, \;\; (u(x.r) - g(x.r)) \geq 0$$

with initial boundary conditions

$$u(x, 0) = g(x, 0)$$

$$u(-x^-, r) = g(-x^-), u(x^+, r) = g(x^+, r) = 0$$

and the conditions that $u(x,r)$ and $\frac{\partial u}{\partial x}(x,r)$ are continuous. The two possibilities in this formulation correspond to situations in which it is optimal to exercise the option $(u = g)$ and those in which it is not $(u > g)$.

We now transform the American call with dividends problem from the original (S,t) variables to (x,r), where x and r refer to the following transformation:

$$x = \ln\left(\frac{S}{E}\right)$$
$$r = \frac{\sigma^2}{2}(T-t)$$
$$v(x,r) = \frac{C}{E}$$
$$v = e^{\alpha x + \beta r} u(x,r)$$

for $-\infty < x < \infty, r > 0$ where $\alpha = -\frac{1}{2}(k'-1), \beta = -\frac{1}{4}(k'-1) - k, k' = \frac{r-d}{\sigma^2/2}$ and $k = \frac{r}{\sigma^2/2}$. Multiplying $v(x,r)$ by E gives us the the value of the option with respect to variables S and t. Through this transformation , the Black Scholes equation becomes

$$\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ r > 0$$

This pertains to a diffusion equation that is characterized by an accompanying initial condition.

$$u(x,0) = v(x,0)e^{\alpha x} = \max(e^{\frac{1}{2}(k\prime+1)} - e^{\frac{1}{2}(k\prime-1)x}, 0)$$

In the transformation given, it can be observed that the dividend component is not explicitly present in the equation. Consequently, the challenge of determining the optimal exercise boundary $x_f(r)$ and the corresponding $u(x,r)$ for American call options that include dividends is equivalent to solving the obstacle problem.

$$\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial x^2} \ \text{ for } x \leq x_f(r)$$
$$u(x,r) = g(x,r) \text{ for } x > x_f(r)$$

with boundary conditions

$$u(x,0) = g(x,0)$$

$$\lim_{x\to\infty} u(x,r) = g(x,r)$$
$$\lim_{x\to-\infty} u(x,r) = 0$$

where

$$g(x,r) = e^{(\frac{1}{4}(k'-1)^2+k)r}\max(e^{\frac{1}{2}(k'+1)} - e^{\frac{1}{2}(k'-1)x}, 0) \tag{3.2.18}$$

We also have the constraint

$$u(x,r) \geq g(x,r) \tag{3.2.19}$$

As we will be utilizing finite difference methods to obtain numerical solutions, we will limit our analysis to a finite interval. Specifically, we will consider the interval $[x^-, x^+]$, where x^- and x^+ represent large negative and positive values, respectively. Consequently, the boundary conditions for the problem are defined as follows:

$$u(x^-, r) = 0, u(x^+, r) = g(x^+, r) \tag{3.2.20}$$

Now rewriting in a linear complementary form , we obtain

$$\left(\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,r) - g(x,r)) = 0 \tag{3.2.21}$$

with two constraints

$$\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial x^2} \geq 0,\ u(x,r) - g(x,r) \geq 0 \tag{3.2.22}$$

with the initial condition $u(x,r) = g(x,0)$ and the boundary conditions

$$u(x^-, r) = g(x^-, r) = 0$$
$$u(x^+, r) = g(x^+, r) \tag{3.2.23}$$

The transformation outlined above is advantageous as it simplifies the diffusion equation and reduces the complexity present in the Black-Scholes equation. It is easier to obtain numerical solutions for the simplified diffusion equation and subsequently convert them to numerical solutions for the Black-Scholes equation using a change of variables, as opposed to directly solving the Black-Scholes equation. Therefore, our strategy for determining the numerical value of American call options involves solving the following equation:

$$\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{3.2.24}$$

and make sure

$$u(x, r) - g(x, r) \geq 0 \tag{3.2.25}$$

3.3 Numerical Methods

The researcher aimed to address the challenge of solving the obstacle problem presented by American options using Finite Difference Methods. To achieve this, the researcher utilized three different finite difference methods namely: the Forward Difference Method (also known as the Explicit method), the Backward Difference Method (also known as the Implicit method), and the Crank Nicolson method. The purpose of using these methods was to find a numerical solution to the obstacle problem, which arises from the early exercise feature of American options. These numerical schemes are created by discretising the continuous equation that describe the option's value over time and the underlying asset's price. The resulting system of equations is then solved using numerical methods, and the accuracy of the resulting estimates is examined to determine the validity of each method. To implement and solve these numerical schemes, the researcher used a programming language called Python and its associated packages. Python is a popular language for scientific computing that provides many tools and libraries for numerical analysis and optimization. With the help of Python packages such as NumPy, SciPy, and QuantLib, the numerical schemes can were efficiently implemented and solved.

Chapter 4

Numerical Methods for American Options

In the previous chapter, a modified Black-Scholes formula was presented for valuing American options. This formula was subsequently transformed into a diffusion equation, utilizing the linear complementarity transformation, which is suitable for numerical methods. This chapter will focus on utilizing the finite difference method to solve the valuation problem posed by the American options. The numerical techniques used in this analysis will be rigorously examined, and a solution for the valuation problem of American options will be derived and analysed.

4.1 Finite Difference Methods

Finite difference methods are a methodology for solving partial differential equations and linear complementarity problems. They are a powerful and adaptable tool that can yield accurate numerical solutions for all of the models discussed in this thesis when utilized correctly. The main idea behind the finite difference approach is to use Taylor series expansions of functions near the locations of interest to approximate the partial derivatives in partial differential equations. This research will focus on three types of Finite Difference Methods namely the Backward Difference Method , the Forward Difference Method and the Crank Nicolson Method.

In the preceding chapter, a method was presented for simplifying the free boundary problem into a fixed problem that can be solved. This approach utilizes the linear complementarity transformation, which enables the compact representation of the American option valuation problem in linear transformation form. A significant advantage of this approach is that the free boundary is not explicitly involved. By solving the linear complementarity problem, the free boundary can be determined using the condition that defines it. This approach is equally applicable in cases where there are multiple boundaries or none at all. The free boundaries can be identified at the points where the first intersection of the function $u(x,r)$ with the function $g(x,r)$ occurs.

4.2 The Forward Difference Method

The forward difference method is a numerical method used to approximate the derivatives of a function at a particular point. It is a type of finite difference method, which means that it approximates derivatives by using a finite difference approximation. Specifically, the forward difference method approximates the derivative of a function at a point by computing the slope of a line that passes through another point.

Now using a forward difference to approximate $\frac{\partial u}{\partial r}$ and a second order central difference $\frac{\partial^2 u}{\partial x^2}$. Thus we can rewrite the reduced form of the as

$$\frac{u^{m+1} - u_n^m}{\delta r} + O(\delta r) = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} + O((\delta x)^2) \tag{4.2.1}$$

Ignoring the terms of $O(\delta r)$ and $O((\delta x)^2)$,we can rearrange the above equation to give the difference equation

$$u_n^{m+1} = \alpha u_{n+1}^m + (1 - 2\alpha)u_n^m + \alpha u_{n-1}^m \tag{4.2.2}$$

where α is given by $\alpha = \delta r/(\delta x)^2$. At the time step of $m+1$, we have the knowledge of the values of u_n^m for all n. Therefore, we can calculate u_n^{m+1} explicitly using the equation mentioned above. In order to meet the constraint of $u(x,r) - g(x,r) \geq 0$, the numerical scheme for the forward difference method can be represented as follows:

$$u_n^{m+1} = \alpha u_{n+1}^m + (1 - 2\alpha)u_n^m + \alpha u_{n-1}^m \tag{4.2.3}$$
$$u_n^{m+1} = \max(y_n^{m+1}, g_n^{m+1})$$

The constraint ensures that the numerical solution obtained satisfies the physical condition that the value of the option cannot be negative. This constraint is necessary because the free boundary represents the boundary between the regions where the option is either exercised or not exercised, and the value of the option is zero in the region where it is not exercised. Therefore, the constraint ensures that the numerical solution is physically meaningful and satisfies the desired properties of the option

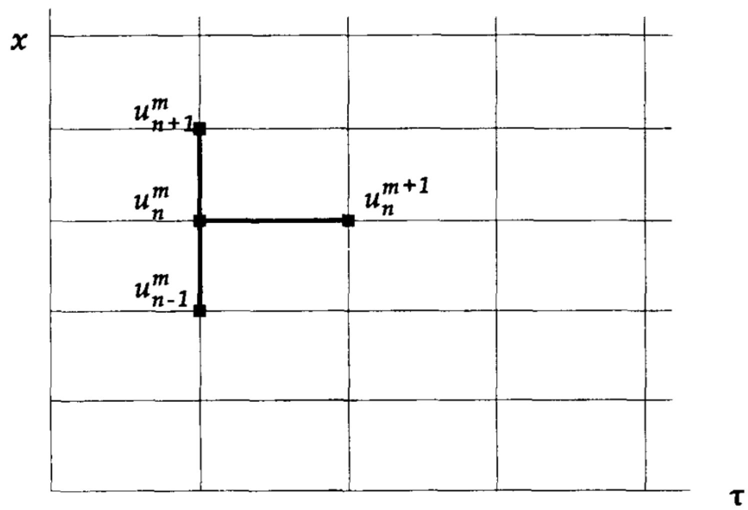


Figure 4.1: Forward Difference Discretization

valuation problem.

In order to obtain the outcomes of the forward difference method, a Python program was employed to solve the finite difference scheme and estimate the value of the American call option. The program establishes a matrix of stock prices and option values, and then proceeds to calculate the option value at each time step using the forward difference method. In the code, the early exercise boundary function is used to determine the nodes in the stock price grid where early exercise would be optimal at each time step. Specifically, the code uses the function to find the index of the first stock price node in the grid that is greater than or equal to the early exercise boundary at each time step. Figure 4.2 shows the results of the code. The output was produced using parameters. *Risk free interest rate* $r = 0.02$ *, Time to maturity* $T = 1$ *,Volatility* $\sigma = 0.15$*, Strike price* $E = 100$*, Number of time steps* $N = 50$*, Number of time steps* $M = 1000$*,* $S_{max} = 300$. Below is the graphical representation of the valuation of American put option using forward difference method.

```
[1.00000000e+02 9.40000000e+01 8.80000000e+01 8.20000000e+01
 7.60000000e+01 7.00000000e+01 6.40000000e+01 5.80000000e+01
 5.20000000e+01 4.60000000e+01 4.00000000e+01 3.40000000e+01
 2.80000000e+01 2.20000000e+01 1.60000000e+01 1.00000000e+01
 4.00000000e+00 1.08097650e-01 1.48603720e-03 1.36687972e-05
 9.43771047e-08 5.21270528e-10 2.39814003e-12 9.45034737e-15
 3.25603366e-17 9.96340547e-20 2.74145180e-22 6.85108473e-25
 1.56797492e-27 3.30932816e-30 6.47938834e-33 1.18287619e-35
 2.02249363e-38 3.25143412e-41 4.93180631e-44 7.07980175e-47
 9.64547103e-50 1.25025573e-52 1.54536867e-55 1.82524175e-58
 2.06388043e-61 2.23809389e-64 2.33129008e-67 2.33604255e-70
 2.25489867e-73 2.09936900e-76 1.88747867e-79 1.64054836e-82
 1.37994341e-85 1.12440361e-88 0.00000000e+00]
```

Figure 4.2: Anerican Put Option Values (Forward Difference Method)

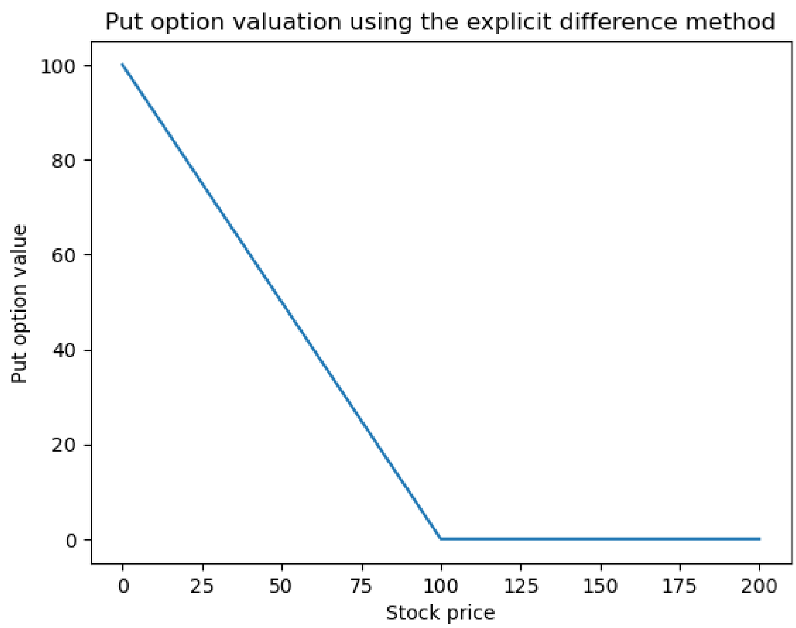


Figure 4.3: American Put Option (Forward Difference Method)

The graph indicates that the forward difference method value is approximately the same as the theoretical value of American option. The forward difference method is easy to implement and computationally efficient. However the forward difference method can suffer from numerical instability when the step size is too large relative to the function being differentiated. This can lead to oscillations or divergent behaviour in the approximation, which can make it difficult to obtain accurate results. It also suffers in the handling of early exercise condition.

The stability of the system is a concern that arises from the above approach. It can be observed that the system is stable when $0 < \alpha \leq \frac{1}{2}$, while it becomes unstable when $\alpha > \frac{1}{2}$. As a result, there are stringent limitations on the size of the time steps that can be employed.

4.3 The Backward Difference Method

The fully implicit method is another term used for the backward difference method. This approach involves determining the solution at a specific time interval by utilizing the solution values at the current and prior time intervals, with the time derivative assessed at the subsequent time interval. The name "backward" is given to this technique because the time derivative is computed at the later time interval, in contrast to the forward difference method where it is computed at the current time interval.

If we apply the backward-difference approximation to $\partial u/\partial r$ and the central difference to $\partial^2 u/\partial x^2$, the resulting transformed equation can be expressed as follows.

$$\frac{u_n^{m+1} - u_n^m}{\delta r} + O(\delta r) = \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\alpha x)^2} + O((\delta x)^2) \tag{4.3.4}$$

Dropping the error terms we get

$$-\alpha u_{n-1}^{m+1} + (1 + 2\alpha)u_n^{m+1} - \alpha u_{n+1}^{m+1} = U_n^m \tag{4.3.5}$$

The reason why the implicit method is named as such is because the updated values cannot be immediately separated and solved for explicitly in relation to the old values. Therefore, an approximation of the linear complementarity problem is utilized.

$$(-\alpha u_{n-1}^{m+1} + (1 + 2\alpha)u_n^{m+1} - \alpha u_{n+1}^{m+1} - u_n^m).(u_n^{m+1} - g_n^{m+1}) = 0 \tag{4.3.6}$$

at time step $m + 1$

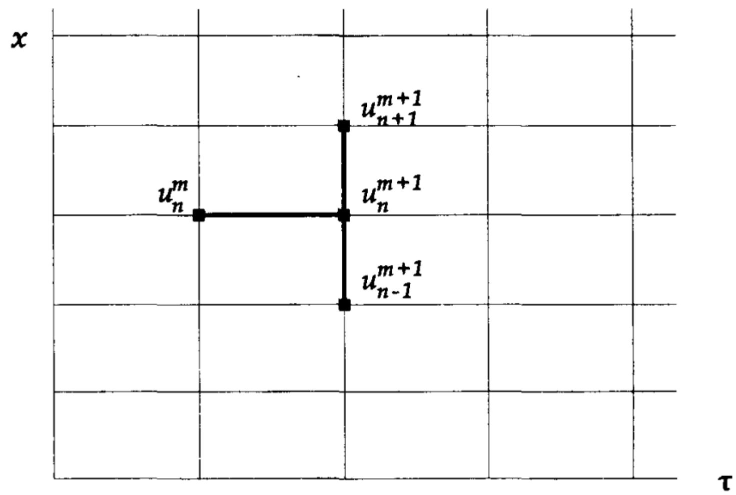


Figure 4.4: Backward Difference Discretization

Let

$$u^m = \begin{pmatrix} u^m_{N^-+1} \\ \cdot \\ \cdot \\ \cdot \\ u^m_{N^+-1} \end{pmatrix}, \; g^m = \begin{pmatrix} g^m_{N^-+1} \\ \cdot \\ \cdot \\ \cdot \\ g^m_{N^+-1} \end{pmatrix} \tag{4.3.7}$$

$$b^m = \begin{pmatrix} b^m_{N^-+1} \\ \cdot \\ \cdot \\ \cdot \\ b^m_o \\ \cdot \\ \cdot \\ \cdot \\ b^m_{N^+-1} \end{pmatrix} = \begin{pmatrix} u^{m-1}_{N^-+1} \\ \cdot \\ \cdot \\ \cdot \\ u^m_o \\ \cdot \\ \cdot \\ \cdot \\ u^{m-1}_{N^+-1} \end{pmatrix} + \alpha \begin{pmatrix} g^m_{N^-+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ g^m_{N^+} \end{pmatrix} \tag{4.3.8}$$

and the coefficient matrix

$$\mathbf{C} = \begin{bmatrix} (1+\alpha) & -\alpha & 0 & \dots & 0 \\ -\alpha & (1+\alpha) & -\alpha & \dots & \vdots \\ 0 & -\alpha & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & (1+\alpha) & -\alpha \\ 0 & \dots & 0 & -\alpha & (1+\alpha) \end{bmatrix} \tag{4.3.9}$$

Thus we want to solve the following constrained matrix problem

$$C\mathbf{u}^{m+1} = \mathbf{b}^{m+1} \tag{4.3.10}$$

In order to ensure that the inequality $u \geq g$ is satisfied, we need to check if it is being met. It is worth noting that the Gerschgorin theorem (Varga 1962) establishes the invertibility of matrix $\mathbf{C}$, which indicates that a solution to the equation exists.

The given all procedures of backward difference method are being implemented in Python to address the American call option valuation problem. The backward difference method is utilized in the code for this purpose. To decide if it is advantageous to exercise the option at an early stage, the early exercise boundary function is defined. The solution update step is altered to compare the solution values inside the boundary with the early exercise boundary, and select the larger value. The solution vector at each time step is then trimmed at the index corresponding to the first underlying asset price value that is equal to or greater than the early exercise boundary. This ensures that the option is exercised early only when it is beneficial to do so. Below is an matrix array showing the value of the American Call value solution.

```
8.00000000e+000 9.00000000e+000 1.00000000e+001 1.10000000e+001
1.20000000e+001 1.30000000e+001 1.40000000e+001 1.50000000e+001
1.60000000e+001 1.70000000e+001 1.80000000e+001 1.90000000e+001
2.00000000e+001 2.10000000e+001 2.20000000e+001 2.30000000e+001
2.40000000e+001 2.50000000e+001 2.60000000e+001 2.70000000e+001
2.80000000e+001 2.90000000e+001 3.00000000e+001 3.10000000e+001
3.20000000e+001 3.30000000e+001 3.40000000e+001 3.50000000e+001
3.60000000e+001 3.70000000e+001 3.80000000e+001 3.90000000e+001
4.00000000e+001 4.10000000e+001 4.20000000e+001 4.30000000e+001
4.40000000e+001 4.50000000e+001 4.60000000e+001 4.70000000e+001
4.80000000e+001 4.90000000e+001 5.00000000e+001 5.10000000e+001
5.20000000e+001 5.30000000e+001 5.40000000e+001 5.50000000e+001
5.60000000e+001 5.70000000e+001 5.80000000e+001 5.90000000e+001
6.00000000e+001 6.10000000e+001 6.20000000e+001 6.30000000e+001
6.40000000e+001 6.50000000e+001 6.60000000e+001 6.70000000e+001
6.80000000e+001 6.90000000e+001 7.00000000e+001 7.10000000e+001
7.20000000e+001 7.30000000e+001 7.40000000e+001 7.50000000e+001
7.60000000e+001 7.70000000e+001 7.80000000e+001 7.90000000e+001
8.00000000e+001 8.10000000e+001 8.20000000e+001 8.30000000e+001
8.40000000e+001 8.50000000e+001 8.60000000e+001 8.70000000e+001
8.80000000e+001 8.90000000e+001 9.00000000e+001 9.10000000e+001
9.20000000e+001 9.30000000e+001 9.40000000e+001 9.50000000e+001
9.60000000e+001 9.70000000e+001 9.80000000e+001 1.02019627e+002
1.04877058e+002]
```

Figure 4.5: American Call Values (Backward Difference Method)

Figure 4:5 shows a python output, showing a valuation matrix after implementing backward difference method with parameters: *Risk free interest rate* $r = 0.05$, *Volatility* $\sigma = 0.2$, *Time to expiration* $T = 1$ *(one year), Strike price* $E = 100$, *Number of time steps* $N = 200$, $S_{max} = 200$, *Number of price steps* $M = 1000$

The rows of the matrix represent the spatial grid points, while the columns represent the time steps. Each element of the solution matrix represents the value of the option at a particular spatial grid point and time step. For example, the element $U[i, j]$ represents the value of the option at the i^{th} spatial grid point and j^{th} time step. The solution matrix can be used to visualize the evolution of the option price over time and space. Figure 4.6 visualizes the solution values using a 3D surface plot, where the x-axis represents time, the y-axis represents the stock price, and the z-axis represents the option price. The plot is coloured using the cool warm colourman to indicate the magnitude of the option price values. The resulting plot shows how the option price changes over time and stock price, taking into account the possibility of early exercise. The 3D surface plot shows the option value as a function of the underlying asset price and time. The x-axis represents time, the y-axis represents the underlying asset price, and the z-axis represents the option value. The surface plot

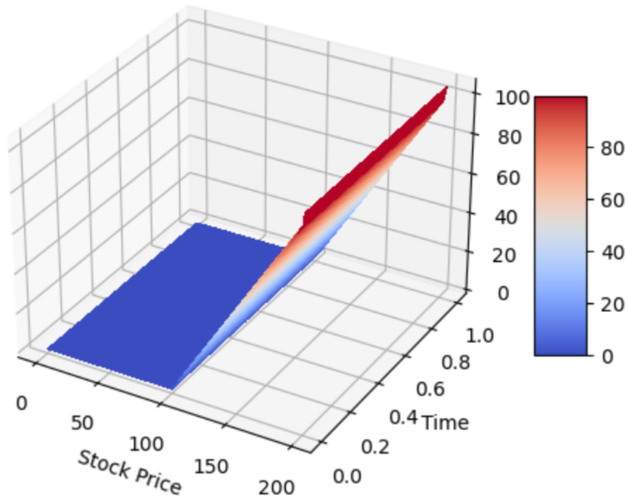


Figure 4.6: American Call Option Value (Backward Difference Method)

is coloured according to a colour-map, with cool colours (e.g. blue) representing low option values and warm colours (e.g. red) representing high option values.

A new code was created to examine whether the method is applicable to different parameters. The focus of this code is to modify the time steps and price steps, which are the primary contributors to fluctuations. The results indicated that the model still produce the same results. Figure 4.7 shows the surface of American call solution value implemented in python under parameters that: *Risk free interest rate* $r = 0.075$, *Volatility* $\sigma = 0.3$, *Strike price* $E = 1000$, *Number of price steps* $N = 300$, *Number of time steps* $M = 2000$, $S_{max} = 3000$, *Time to maturity* $T = 2$ *years*

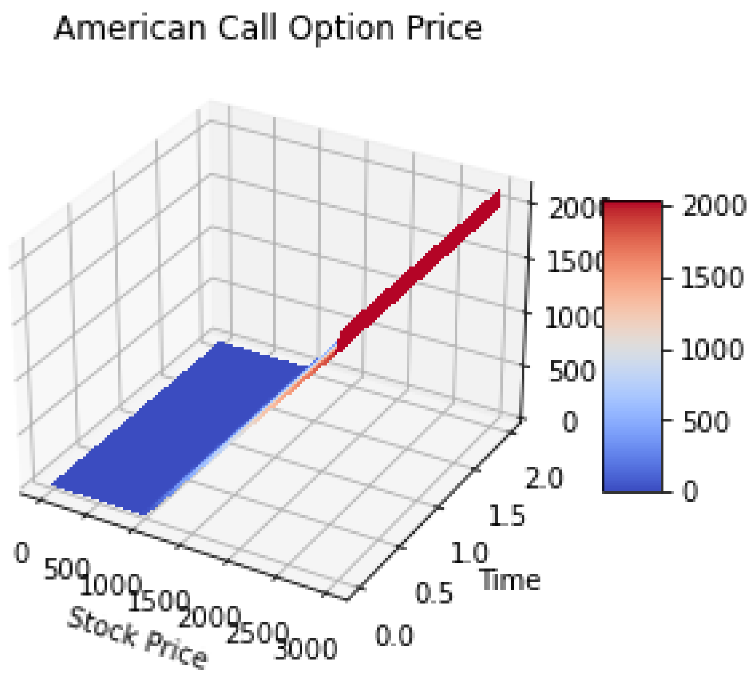


Figure 4.7: American Call Option Price(Backward Difference Method)

Increasing number of time steps M and number of price steps N seems to result in a more accurate solution as indicated by the smoothness of the surface, because it increases the spatial and temporal resolution of the solution. This results in a accurate approximation of the early exercise boundary and the option value. However, increasing M and N will also increase the computational cost of the solution, as the size of the coefficient matrix and the number of iterations required in the fully implicit method will both increase. Therefore, there is a trade-off between the accuracy of the solution and the computational cost.

Now for **American Put Option** applying the backward difference method and solve it using python code. The output of the code is the solution at maturity, which is a vector of size $(N + 1)$, where each element represents the American put option price at a particular stock price. The optimal exercise strategy is implicitly included in the solution, as the put option value is compared with the early exercise boundary at each time step to determine whether to exercise the option early or not.

```
[1.00000000e+003 9.60000000e+002 9.20000000e+002 8.80000000e+002
 8.40000000e+002 8.00000000e+002 7.60000000e+002 7.20000000e+002
 6.80000000e+002 6.40000000e+002 6.00000000e+002 5.60000000e+002
 5.20000000e+002 4.80000000e+002 4.40000000e+002 4.00000000e+002
 3.60000000e+002 3.20000000e+002 2.80000000e+002 2.40000000e+002
 2.00000000e+002 1.60000000e+002 1.20000000e+002 8.00000000e+001
 4.00000000e+001 1.24999883e-005 3.90502473e-009 6.10706335e-013
 6.37391170e-017 4.99439831e-021 3.13391775e-025 1.64038466e-029
 7.36699791e-034 2.89785075e-038 1.01424009e-042 3.19800563e-047
 9.17604656e-052 2.41586983e-056 5.87704411e-061 1.32888934e-065
 2.80726968e-070 5.56517656e-075 1.03937579e-079 1.83514339e-084
 3.07265087e-089 4.89222598e-094 7.42568547e-099 1.07693335e-103
 1.49541297e-108 1.99195837e-113 0.00000000e+000]
```

Figure 4.8: American Put Option Value Matrix

The above matrix was produced with the following parameters, *Risk free interest rate* $r = 0.05$, *Volatility* $\sigma = 0.2$, *Strike price* $E = 1000$, *Time to maturity* $T = 0.5$ *(Six months)* , *Number of time steps* $M = 1000$, *Number of price steps* $N = 50$, $S_{max} = 2000$. Figure 4.9 shows a surface plot of the American put option value, with the stock price and time to maturity on the x and y axes, respectively, and the option value on the z axis. The plot shows how the value of the option changes with changes in the underlying stock price and time to maturity, and correctly reflects the expected behaviour of the put option value. This indicates that the method accurately approximates the value of American put option.

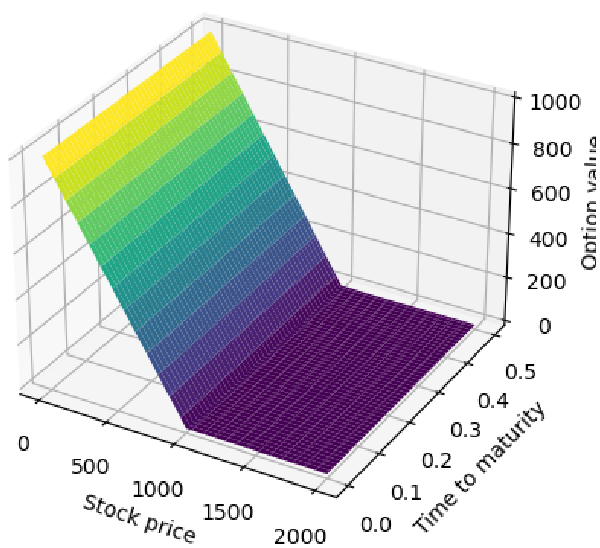


Figure 4.9: American Put Option Value Surface

4.4 The Crank Nicolson Method

The Crank-Nicolson Method provides an enhanced temporal convergence rate compared to the backward difference method. Specifically, it improves the rate from $O(\delta r)$ to $O((\delta)^2)$. To achieve this, the method employs a regular finite mesh on the (x, r) plane and uses finite difference approximations to solve the linear complementarity equations. The approach involves estimating expressions of the form $\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial x^2}$ on the mesh with step sizes of δr and δx, and truncating the values such that x is within the range of $N^- \delta x$ and $N^+ \delta x$.

$$N^- \delta x \leq x = n\delta x \leq N^+ \delta x$$

where N^- and N^+ are suitably large numbers. Now lets consider the Crank -Nicolson method. We see that

$$\frac{\partial u}{\partial r}(x, r + \frac{\delta r}{2}) = \frac{u_n^{m+1} - u_n^m}{\delta r} + O((\delta r)^2)$$

and

$$\frac{\partial u}{\partial r}(x, r + \frac{\delta r}{2}) = \frac{1}{2}\left(\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta x)^2}\right) + \frac{1}{2}\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2}\right) + O((\delta r)^2)$$

where $u_n^m = u(n\delta x, m\delta r)$. Dropping terms of $O((\delta r)^2)$ and $O((\delta x)^2)$the inequality $\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial x^2} \geq 0$ is approximated by

$$u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}) \geq u_n^m + \frac{1}{2}\alpha(u_{n+1}^m - 2u_n^m + u_{n-1}^m) \quad (4.4.11)$$

where alpha is given by

$$\alpha = \frac{\delta r}{(\delta x)^2}$$

we write

$$g_n^m = g(n\delta x, m\delta r) \quad (4.4.12)$$

for the discretised payoff function. The condition $u(x, r) \geq g(x, r)$ is approximated by

$$u_n^m \geq g_n^m \text{ for } m \geq 1 \quad (4.4.13)$$

and the boundary and initial conditions imply that

$$u^m_{N^-} = g^m_{N^-}, \ , u^m_{N^+} = g^m_{N^+} \ , u^0_n = g^0_n \tag{4.4.14}$$

if we define Z^m_n by

$$Z^m_n = (1-\alpha)u^m_n + \frac{1}{2}(u^m_{n+1} + u^m_{n-1}) \tag{4.4.15}$$

equation 4.2.2 then becomes

$$(1+\alpha)u^{m+1}_n - \frac{1}{2}\alpha(u^{m+1}_{n+1} + u^{m+1}_{n-1}) \geq Z^m_n \tag{4.4.16}$$

Note that at time step $(m+1)\delta r$ we can find Z^m_n explicitly , since we know the values of u^m_n. The linear complementarity condition that

$$\left(\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial x^2}\right) \cdot (u(x,r) - g(x,r)) = 0$$

is approximated by

$$\left((1+\alpha)u^{m+1}_n - \frac{1}{2}\alpha(u^{m+1}_{n+1} + u^{m+1}_{n-1} - Z^m_n)\right)(u^{m+1}_n - g^{m+1}_n) = 0 \tag{4.4.17}$$

Now we formulate the finite difference approximations as a constrained matrix problem and describe how the projected SOR method can be used to the problem. Let u^m denote the vector representing the constraint at the same time.

$$u^m = \begin{pmatrix} u^m_{N^-+1} \\ \cdot \\ \cdot \\ \cdot \\ u^m_{N^+-1} \end{pmatrix} , \ g^m = \begin{pmatrix} g^m_{N^-+1} \\ \cdot \\ \cdot \\ \cdot \\ g^m_{N^+-1} \end{pmatrix} \tag{4.4.18}$$

and let the vector b^m be given by

$$
b^m = \begin{pmatrix} b^m_{N^-+1} \\ \cdot \\ \cdot \\ \cdot \\ b^m_o \\ \cdot \\ \cdot \\ \cdot \\ b^m_{N^+-1} \end{pmatrix} = \begin{pmatrix} Z^m_{N^-+1} \\ \cdot \\ \cdot \\ \cdot \\ Z^m_o \\ \cdot \\ \cdot \\ \cdot \\ Z^m_{N^+-1} \end{pmatrix} + \frac{1}{2}\alpha \begin{pmatrix} g^{m+1}_{N^-+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ g^{m+1}_{N^+} \end{pmatrix} \tag{4.4.19}
$$

where the quantities Z_n^m are determined from equation 4.2.6 and the vector on the extreme right end side of 4.3.10 includes the effects of the boundary conditions at $n = N^-$ and $n = N^+$. Thus we introduce tri-diagonal, symmetric matrix

$$
\mathbf{C} = \begin{bmatrix} 1+\alpha & -\frac{1}{2}\alpha & 0 & \dots & 0 \\ -\frac{1}{2}\alpha & 1+\alpha & -\frac{1}{2}\alpha & \dots & \vdots \\ 0 & -\frac{1}{2}\alpha & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1+\alpha & -\frac{1}{2}\alpha \\ 0 & \dots & 0 & -\frac{1}{2}\alpha & 1+\alpha \end{bmatrix} \tag{4.4.20}
$$

Thus we can rewrite our discrete approximation to the linear complementarity problem in matrix form as

$$
\begin{gathered} Cu^{m+1} \geq b^m \ , \ u^{m+1} \geq g^{m+1} \\ (u^{m+1} - g^{m+1}) \cdot (Cu^{m+1} - b^m) = 0 \end{gathered} \tag{4.4.21}
$$

When using vectors a and b, we interpret the statement $a \geq b$ to mean that each element of vector a is greater than or equal to its corresponding element in vector b, i.e., $a_n \geq b_n$ for all n. In the scheme's time-stepping process, the vector b^m contains information from time step $m\delta r$ that determines the value of v^{m+1} at time step $(m+1)\delta r$. We can calculate b^m at each time-step using the known values of v^m. To solve problem 4.3.12 at any time step, we can calculate g^m for any $m\delta r$ using the modified form of SOR, known as the projected SOR.

4.5 The Projected SOR

The SOR method can be slightly modified to suit a Crank-Nicolson finite difference formulation of the problem, resulting in the projected SOR equations shown below. These equations can be used to solve for $Cu^{m+1} = b^m$ subject to the constraint that $u^{m+1} \geq g^{m+1}$ by iterating until $u_n^{m+1,k}$ converge to u_n^{m+1}:

$$\begin{aligned} y_n^{m+1,k+1} &= \frac{1}{1+\alpha}\left(b_n^m + \frac{1}{2}\alpha(u_{n-1}^{m+1,k+1} + u_{n+1}^{m+1,k})\right) \\ u_n^{m+1,k+1} &= u_n^{m+1,k} + w(y_n^{m+1,k+1} - u_n^{m+1,k}) \end{aligned} \tag{4.5.22}$$

To ensure the constraint of $u^{m+1} \geq g^{m+1}$ is met, we need to modify the second equation accordingly.

A Python code was utilized to apply the Crank Nicolson method for resolving the valuation of American Call Option. The code then constructs the coefficient matrix for the Crank-Nicolson method and initializes the solution matrix. It uses a loop to iterate through time steps and solve for the option value at each step. The loop uses the Crank-Nicolson method to solve the problem, and includes a condition to enforce the early exercise boundary. Finally, the code prints the option value at maturity. An output generated by the code is presented in Figure 4.10.

```
d1 = (np.log(S/K) + (r + 0.5*sigma**2)*(T - t)) / (sigma*np.sqrt

[0.00000000e+00 1.12187646e-80 1.11118149e-78 1.08892819e-76
 1.04539000e-74 9.82728931e-73 9.04209848e-71 8.13916797e-69
 7.16394288e-67 6.16256886e-65 5.17815190e-63 4.24761991e-61
 3.39951404e-59 2.65288151e-57 2.01727084e-55 1.49367578e-53
 1.07616373e-51 7.53870348e-50 5.13048561e-48 3.38914434e-46
 2.17117520e-44 1.34756998e-42 8.09488049e-41 4.70102998e-39
 2.63624964e-37 1.42574385e-35 7.42621296e-34 3.71987059e-32
 1.78910177e-30 8.24790117e-29 3.63782687e-27 1.53195275e-25
 6.14586488e-24 2.34307709e-22 8.46592337e-21 2.89022863e-19
 9.29159747e-18 2.80216834e-16 7.89345691e-15 2.06663565e-13
 5.00043781e-12 1.11073646e-10 2.24730229e-09 4.10272268e-08
 6.68132083e-07 9.56838915e-06 1.18334509e-04 1.23403101e-03
 1.05050334e-02 6.97148090e-02 3.36835024e-01 1.06971481e+00
 2.01050503e+00 3.00123403e+00 4.00011833e+00 5.00000957e+00
 6.00000067e+00 7.00000004e+00 8.00000000e+00 9.00000000e+00
 1.00000000e+01 1.10000000e+01 1.20000000e+01 1.30000000e+01
 1.40000000e+01 1.50000000e+01 1.60000000e+01 1.70000000e+01
 1.80000000e+01 1.90000000e+01 2.00000000e+01 2.10000000e+01
 2.20000000e+01 2.30000000e+01 2.40000000e+01 2.50000000e+01
 2.60000000e+01 2.70000000e+01 2.80000000e+01 2.90000000e+01
 3.00000000e+01 3.10000000e+01 3.20000000e+01 3.30000000e+01
 3.40000000e+01 3.50000000e+01 3.60000000e+01 3.70000000e+01
 3.80000000e+01 3.90000000e+01 4.00000000e+01 4.10000000e+01
 4.20000000e+01 4.30000006e+01 4.40000085e+01 4.50001068e+01
 4.60011294e+01 4.70098040e+01 4.80669972e+01 4.93395427e+01
 5.24385288e+01]
```

Figure 4.10: American Call option Value (Crank Nicolson Method)

The output of the code is the option value at maturity, which is a vector containing the option value for each possible stock price. This output was produced with parameters defined as: *Initial stock price* $S_0 = 50$, *Volatility* $\sigma = 0.2$, *Time to maturity* $T = 1$, *Exercise price* $E = 50$, $S_{min} = 0$, $S_{max} = 100$, *Number of price steps* $N = 100$, *Number of time steps* $M = 100$.

Now the same code was modified to solve for American put option. The modifications made to the code include changing the payoff function to a put option, adjusting the early exercise boundary function accordingly, and changing the initial and boundary conditions to reflect the put option's payoff structure. The results are presented in figure 4.11

```
[5.23176183e+01 5.40000000e+01 5.30000000e+01 5.20000000e+01
 5.10000000e+01 5.00000000e+01 4.90000000e+01 4.80000000e+01
 4.70000000e+01 4.60000000e+01 4.50000000e+01 4.40000000e+01
 4.30000000e+01 4.20000000e+01 4.10000000e+01 4.00000000e+01
 3.90000000e+01 3.80000000e+01 3.70000000e+01 3.60000000e+01
 3.50000000e+01 3.40000000e+01 3.30000000e+01 3.20000000e+01
 3.10000000e+01 3.00000000e+01 2.90000000e+01 2.80000000e+01
 2.70000000e+01 2.60000000e+01 2.50000000e+01 2.40000000e+01
 2.30000000e+01 2.20000000e+01 2.10000000e+01 2.00000000e+01
 1.90000000e+01 1.80000000e+01 1.70000000e+01 1.60000000e+01
 1.50000000e+01 1.40000000e+01 1.30000000e+01 1.20000000e+01
 1.10000000e+01 1.00000000e+01 9.00000000e+00 8.00000000e+00
 7.00000004e+00 6.00000067e+00 5.00000957e+00 4.00011833e+00
 3.00123403e+00 2.01050503e+00 1.06971481e+00 3.36835024e-01
 6.97148090e-02 1.05050334e-02 1.23403101e-03 1.18334509e-04
 9.56838915e-06 6.68132083e-07 4.10272268e-08 2.24730229e-09
 1.11073646e-10 5.00043781e-12 2.06663565e-13 7.89345691e-15
 2.80216834e-16 9.29159747e-18 2.89022863e-19 8.46592337e-21
 2.34307709e-22 6.14586488e-24 1.53195275e-25 3.63782687e-27
 8.24790117e-29 1.78910177e-30 3.71987059e-32 7.42621296e-34
 1.42574385e-35 2.63624964e-37 4.70102998e-39 8.09488049e-41
 1.34756998e-42 2.17117520e-44 3.38914434e-46 5.13048561e-48
 7.53870348e-50 1.07616373e-51 1.49367578e-53 2.01727084e-55
 2.65288151e-57 3.39951404e-59 4.24761991e-61 5.17815190e-63
 6.16256886e-65 7.16394288e-67 8.13917818e-69 9.13811166e-71
 0.00000000e+00]

C:\Users\talen\AppData\Local\Temp\ipykernel_10984\2918719004.py:26:
```

Figure 4.11: American Put Option Values (The Crank Nicolson Method)

Figure 4.9 shows the output of an American put option produced parameters: *Initial stock price* $S_0 = 50$, *Volatility* $\sigma = 0.2$, *Risk free interest rate* $r = 0.5$, *Time to maturity* $T = 1$, *Exercise price* $E = 55$, $S_{min} = 0$, $S_{max} = 100$, *Number of price steps* $N = 100$, *Number of time steps* $M = 1000$.

One of the observation discovered from Crank-Nicolson is that it can handle the variable coefficient in the PDE that arises from the possibility of early exercise. Other numerical methods, such as the explicit finite-difference method require a different PDE to account for early exercise, which can be more complicated and computationally expensive to solve.

4.6 The Determination of the Optimal Exercise Boundary for American-style Options

In the previous section, we discussed three Finite Difference Methods, all of which have algorithms designed to determine the early exercise boundary. This was one of the main objectives of the thesis. In this section, we will provide a brief overview of how the early exercise boundary was incorporated into the difference scheme.

The Partial Differential Equation (PDE) that governs the option value was solved using the Projected Successive Over Relaxation (PSOR) method. It should be noted that this method does not offer direct insight into the positions of the early exercise boundaries $S_f(t)$. Nevertheless, the value of $S_f(t)$ can be determined retrospectively from the solution $V(S,t)$.

Assuming that the solution $V(S,t)$ is already known for t belonging to the interval $[0,T]$, the objective is to determine the early exercise position $S_f(t)$. For a call option, $S_f(t)$ represents the smallest value of S for which $V(S,t) = S - E$, indicating that the intrinsic value of the option is equivalent to the payoff from immediate exercise. Conversely, for a put option, $S_f(t)$ corresponds to the largest value of S such that $V(S,t) = E - S$

To determine $S_f(t)$ through numerical methods, a numerical approximation is utilized to yield the values of the approximate solution at nodal points $(S_i, T - r_j)$, where $S_i = Ee^{ih}$ and $r_j = j_k$. Subsequently, the following algorithm is employed to establish the location of the early exercise boundary:

For a call option:
Iterate over all nodal points $(S_i, T - r_j)$. For each nodal point, check if $|V(S_i, T - r_j) - (S_j - E)| < \epsilon$, where ϵ is a prescribed tolerance level. If the condition in step 2 is satisfied, then set $S_f(t)$ to S_i.

For a put option:
Iterate over all nodal points $(S_i, T - r_j)$. For each nodal point, check if $|V(S_i, T - r_j) - (E - S_i)| < \epsilon$, where ϵ is a prescribed tolerance level. If the condition in step 2 is satisfied, then set $S_f(t)$ to S_i. In other words, we iterate over all nodal points and look for the first point that satisfies the condition for the early exercise boundary. The tolerance level ϵ controls the accuracy of the approximation.

Chapter 5

Conclusions and Recommendations

5.1 Conclusion

In conclusion this dissertation places particular emphasis on the pricing of American-style options. Initially, efforts were directed towards deriving the Black-Scholes formula based on a set of assumptions. It is worth noting that the formula gives a solution to closed form options such as the European options. The formula was then extended to fit the conditions of American options which permits the owner of option to exercise the option at any time before the maturity date. It was observed that American option valuation results in an inequality instead of an equation. The formula postulates the free boundary problem which can be interpreted as the classical obstacle problem. The analogue between the American option valuation problem and the obstacle problem was then presented to formulate the Linear complementarity transformation. This through change of variables reduces the problem into a fixed boundary problem under which the free boundary problem can be inferred afterwards.

Due to the early exercise potential of American-style options, the exercise boundary remains variable. Since there are no closed-form solutions available for solving the inequality in the pricing function, numerical approximations were employed. Three Finite difference schemes were studied, namely the forward difference method (Explicit method), backward difference method (implicit method), and Crank-Nicolson. The backward difference method method is the easiest to program, but it may generate unstable results unless stability constraints are imposed. The implicit scheme, on the other hand, is more challenging to program, but it provides stable results regardless of the mesh density. Finally, the Crank-Nicolson method takes longer to compute a bit more complicated to program but is the most accurate approach when it is applied correctly as it combines the advantages of both the forward difference and backward difference schemes.

The research has demonstrated that finite difference methods offer an effective means of computing option values, particularly for problems where closed-form solutions are not available. However, it is important to consider the limitations and potential instability of these methods, particularly when employing excessively large time steps.

5.2 Recommendations

Based on the research conducted on the valuation of American-style options using finite difference methods, the following recommendations are suggested:

1. **Refine the model assumptions**: The accuracy and stability of finite difference methods can be improved by refining the assumptions used in the model. Future research should focus on identifying and addressing any unrealistic assumptions that may impact the accuracy of the results.

2. **Enhance accuracy through adaptive mesh refinement**: Adaptive mesh refinement techniques can be used to improve the accuracy of finite difference methods by refining the mesh density in areas where the solution is changing rapidly. Future research should investigate the effectiveness of adaptive mesh refinement for pricing American options.

3. **Apply to finite difference other areas of finance**: Finite difference methods have been widely used in finance for pricing mainly options, but there is a need to apply these methods to other areas in finance where complex problems are not easily solvable explicitly. Future research should focus on applying finite difference methods to other problems, such as bond valuation and credit rsik modelling.

In summary it is recommended that future research should focus on enhancing the accuracy and stability of finite difference methods by exploring alternative numerical techniques and refining the model assumptions. Additionally, it is suggested that further investigation should be conducted into the application of these methods to more complex areas in finance such as those of credit risk modelling and bond valuation problems. Overall, the application of finite difference methods to American option valuation presents a promising area for future research and can have significant implications for the wider financial industry.

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Appendix A

A.1 Forward Difference Method Python Code

```
import numpy as np

# Define parameters
r = 0.02
sigma = 0.15
K = 100
T = 1
Smax = 300
N = 50
M = 1000
dt = T / M
ds = Smax / N
s = np.arange(0, Smax + ds, ds)
t = np.arange(0, T + dt, dt)
alpha = dt / (ds**2)

# Define the payoff function for a put option
def put_payoff(S, K):
return np.maximum(K - S, 0)

# Define the early exercise boundary function
def early_exercise_boundary(t, S, K, r, sigma):
return K - (S - K) * np.exp(-r * (T - t)) + sigma * np.sqrt(t) * np.sqrt(T
 - t) *
 np.sqrt(2 / np.pi)
```

```
# Define the initial and boundary conditions

u = put_payoff(s, K)
u[0] = K
u[N] = 0

# Use the explicit difference method to solve the problem
for m in range(1, M + 1):
u[1:N] = (1 - 2*alpha)*u[1:N] + alpha*u[2:N+1] + alpha*u[0:N-1]
u[0] = K
u[N] = 0
u[:np.argmax(s >= early_exercise_boundary(t[m], s, K, r, sigma))] =
 put_payoff(s[:np.argmax(s >= early_exercise_boundary(t[m], s, K, r,
  sigma))], K)

# Print the solution at maturity
print(u)
```

A.2 Backward Difference Method Python Code

```
import numpy as np

# Define parameters
r = 0.075
sigma = 0.3
K = 1000
T = 1.5
Smax = 3000
N = 200
M = 2000
```

```
dt = T / M
ds = Smax / N
s = np.arange(0, Smax + ds, ds)
t = np.arange(0, T + dt, dt)
alpha = dt / (ds**2)

# Define the payoff function for a call option
def call_payoff(S, K):
return np.maximum(S - K, 0)

# Define the early exercise boundary function
def early_exercise_boundary(t, S, K, r, sigma):
return K + (S - K) * np.exp(-r * (T - t)) - sigma * np.sqrt(t) *
 np.sqrt(T - t) * np.sqrt(2 / np.pi)

# Define the initial and boundary conditions
u = call_payoff(s, K)
u[0] = 0
u[N] = Smax - K * np.exp(-r * T)

# Define the coefficient matrix
C = np.diag(1 + 2 * alpha * np.ones(N - 1)) - np.diag(alpha * np.ones(N
 - 2), -1) - np.diag(alpha * np.ones(N - 2), 1)

# Define the solution matrix
U = np.zeros((N + 1, M + 1))
U[:, 0] = u

# Use the fully implicit method to solve the problem
for m in range(1, M + 1):
b = np.zeros(N - 1)
b[0] = alpha * (u[0] + u[1])
```

```
b[-1] = alpha * (u[-1] + u[-2])
b = b + u[1:N] - alpha * call_payoff(s[1:N], K)
u_interior = np.linalg.solve(C, b)
u[1:N] = np.maximum(u_interior, call_payoff(s[1:N], K))
u[:np.argmax(s >= early_exercise_boundary(t[m], s, K, r, sigma))] =
 call_payoff(s[:np.argmax(s >= early_exercise_boundary(t[m], s, K, r,
  sigma))], K)
U[:, m] = u

# Print the solution at maturity
print(U[:, -1])
```

A.3 The Crank-Nicolson Method Python Code

```
import numpy as np
from scipy.stats import norm

# Define parameters
S0 = 50
K = 50
r = 0.05
sigma = 0.2
T = 1
Smin = 0
Smax = 100
N = 100
M = 1000
dt = T / M
ds = Smax / N
s = np.arange(Smin, Smax + ds, ds)
t = np.arange(0, T + dt, dt)
alpha = dt / (2*ds**2)
```

```
# Define the payoff function for an American call option
def call_payoff(S, K):
return np.maximum(S - K, 0)

# Define the early exercise boundary function
def early_exercise_boundary(t, S, K, r, sigma):
d1 = (np.log(S/K) + (r + 0.5*sigma**2)*(T - t)) / (sigma*np.sqrt(T - t))
return S - K*np.exp(-r*(T - t))*norm.cdf(d1)

# Define the initial and boundary conditions
u = call_payoff(s, K)
u[0] = 0
u[N] = Smax - K*np.exp(-r*T)

# Define the coefficient matrix
C = np.diag(1 + alpha * np.ones(N - 1)) - 0.5 * alpha * np.diag(np.ones(N -
 2), -1) - 0.5 * alpha * np.diag(np.ones(N - 2), 1)
D = np.diag(1 - alpha * np.ones(N - 1)) + 0.5 * alpha * np.diag(np.ones(N -
 2), -1) + 0.5 * alpha * np.diag(np.ones(N - 2), 1)

# Define the solution matrix
U = np.zeros((N + 1, M + 1))
U[:, 0] = u

# Use the Crank-Nicolson method to solve the problem
for m in range(1, M + 1):
b = D.dot(u[1:N])
b[0] += 0.5*alpha*(2*u[1] + (r - 0.5*sigma**2)*ds*u[0])
b[-1] += 0.5*alpha*((r + 0.5*sigma**2)*ds*u[N] + 2*u[N-1])
u_interior = np.linalg.solve(C, b)
u[1:N] = np.maximum(u_interior, call_payoff(s[1:N], K))
```

```
u[:np.argmax(s >= early_exercise_boundary(t[m], s, K, r, sigma))] =
 call_payoff(s[:np.argmax(s >= early_exercise_boundary(t[m], s, K, r,
  sigma))], K)
U[:, m] = u

# Print the option value at maturity
print(U[:, -1])
```